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THE STRICT DUAL OF B^*-ALGEBRAS

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ABSTRACT. If A is a closed two-sided ideal in the B^*-algebra X, then $(X, \beta_A)^*$ with the strong topology is isomorphic to A^*, where β_A is the strict topology on X.

Introduction. Let $C_b(X)$ be the Banach space of all real-valued bounded continuous functions on X. It is well known that if X is compact then $(C_b(X), \sigma)^*$ is isomorphic to $M(X)$ where $M(X)$ is the bounded Radon measure on X and σ is the usual topology of uniform convergence. However, if X is only locally compact Hausdorff then $(C_b(X), \sigma)^*$ is isomorphic to $M(\beta X)$ where βX is the Stone-Čech compactification of X. In 1958, R. C. Buck [1] showed that if $C_b(X)$ is given the strict topology generated by $C_0(X)$, denoted as the β topology, then $(C_b(X), \beta)^*$ is isomorphic to $M(X)$. We show that this result is essentially due to the fact that $C_0(X)$ is a closed two-sided ideal of $C_b(X)$.

1. DEFINITION. Let X be a B^*-algebra and A a closed two-sided ideal in X. The strict topology of X with respect to A, denoted as the β_A topology on X, is the locally convex topology generated by the seminorms $(\lambda_a)_{a\in A}$ and $(\rho_a)_{a\in A}$ where $\lambda_a(x) = \|ax\|$ and $\rho_a(x) = \|xa\|$. We will denote X with the β_A topology by the pair (X, β_A). It is clear that β_A is a vector space topology in which multiplication is separately continuous.

2. PROPOSITION. *Let A be a closed two-sided ideal in the B^*-algebra X. Then A is β_A dense in X.*

PROOF. C. E. Rickart [3] has shown that A is an invariant subspace with respect to involution and, hence, A can be considered as a B^*-algebra. If $\{e_\alpha | \alpha \in \Lambda\}$ is an approximate identity in A [3, p. 245], then $e_\lambda x + xe_\lambda - e_\lambda x e_\lambda$ converges to x in the β_A topology for each $x \in X$. Since A is a two-sided ideal, the set $\{e_\lambda x + xe_\lambda - e_\lambda x e_\lambda | \lambda \in \Lambda\}$ is contained in A and, hence, the result follows.

3. PROPOSITION. *Let A be a closed two-sided ideal in the B^*-algebra X. Then*

$$(X, \beta_A)^* = \{f \cdot a | a \in A, f \in X^*\}.$$

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PROOF. Let $f \in (X, \beta_A)^*$ and ϕ be the inclusion mapping of A into X. Then $f\phi \in A^*$. Since A is a B^*-algebra there exists an $a \in A$ and $g \in A^*$ such that $f\phi = g \cdot a$ where $g \cdot a(x) = g(ax)$ [5]. By the Hahn-Banach theorem g can be extended to an $h \in X^*$ such that $g = h\phi$. Let $\{e_\alpha\}$ be an approximate identity for A. Since $\{e_\alpha x + xe_\alpha - e_\alpha x e_\alpha\}$ converges to x in the β_A topology and A is a closed two-sided ideal, we have that

$$\begin{aligned} f(x) &= \lim_\alpha f(e_\alpha x + xe_\alpha - e_\alpha x e_\alpha) \\ &= \lim_\alpha g \cdot a(e_\alpha x + xe_\alpha - e_\alpha x e_\alpha) = g(ax) \\ &= h(ax) = h \cdot a(x). \end{aligned}$$

To get the reverse inclusion, it is sufficient to observe that if x_α converges to x in the β_A topology and $a \in A$, then ax_α converges to ax in the norm topology. Thus, for $h \in X^*$, $h \cdot a(x_\alpha)$ converges to $h \cdot a(x)$ and, therefore, $h \cdot a$ is β_A continuous.

4. LEMMA. *Let A be a closed two-sided ideal in the B^*-algebra X and S a β_A bounded subset of X. Then* $\sup\{\|xa\|: x \in S, a \in A, \|a\| < 1\}$ *is finite.*

PROOF. For each $x \in S$, define T_x from A into A by $T_x a = xa$. The map T_x is clearly linear and continuous. Since S is a β_A bounded subset, for each $a \in A$, there exists an $M(a) > 0$ such that $\sup_{x \in S}\{\|T_x a\| = \|xa\|\} < M(a)$. Hence, by the uniform boundedness principle, there exists an $M > 0$ such that $\|T_x\| < M$ for all $x \in S$. But

$$\|T_x\| = \sup_{\|a\|<1} \|T_x a\| = \sup\{\|xa\|: a \in A, \|a\| < 1\}$$

and, hence, the Lemma follows.

The strong topology on $(X, \beta_A)^*$ is defined to be the topology of uniform convergence on the β_A bounded subsets of (X, β_A). Using this topology, we have the following result.

5. THEOREM. *If A is a closed two-sided ideal in the B^*-algebra X, then $((X, \beta_A)^*, \gamma)$ is isomorphic to A^*, where γ is the strong topology on $(X, \beta_A)^*$.*

PROOF. From Proposition 2 we have that A is β_A dense in X. Let ϕ be the inclusion mapping of A into X and let ϕ^* be the adjoint mapping of $(X, \beta_A)^*$ into A^*. We now show that ϕ^* is an isomorphism.

Since A is β_A dense in X, it follows that ϕ^* is one-to-one. Let $f \in A^*$. Then $f = g \cdot a$ for some $a \in A$ and $g \in A^*$ where $g \cdot a(x) = g(ax)$ [5]. By the Hahn-Banach theorem, we can extend g to all of X to obtain $\hat{g} \in X^*$ such that $\hat{g}|_A = g$. Since $\phi^*(\hat{g} \cdot a)(b) = \hat{g} \cdot a(\phi(b)) = g \cdot a(b)$ for all $b \in A$, it follows that ϕ^* is onto.

The continuity of ϕ^* is true since norm bounded sets are also β bounded sets. To show that ϕ^{*-1} is continuous, it is sufficient to show that if $\{f_n\}$ is a sequence in A^* that converges to $f \in A^*$, then $\phi^{*-1}(f_n) = F_n$ converges in the strong topology to $\phi^{*-1}(f) = F$.

Let S be a β_A bounded set. By Lemma 4, there exists a number $M > 0$ such that

$$\sup\{\|xa\| : x \in S, a \in A \text{ and } \|a\| < 1\} < M.$$

Let $\varepsilon > 0$. Since $\{f_n\}$ converges to f there exists an N such that for all $n > N$, $\|f_n - f\| < \varepsilon/3M$. Since $\{f_n | n \in Z^+\} \cup \{f\}$ is totally bounded, for each $k > 0$, there exists $a_k \in A$, $g_k \in A^*$ and $g_{nk} \in A^*$ satisfying

(5.1) $f_n = g_{nk} \cdot a_k$ and $f = g_k \cdot a_k$,

(5.2) $\|f_n - g_{nk}\| < 1/k$ and $\|f - g_k\| < 1/k$,

(5.3) $\|a_k\| < 1$ [4].

Now $F_n = G_{nk} \cdot a_k$ and $F = G_k \cdot a_k$, where G_{nk} is a Hahn-Banach extension of g_{nk} and G_k is a Hahn-Banach extension of g_k. We choose k such that $1/k < \varepsilon/3M$. The continuity of ϕ^{*-1} now follows from

$$\begin{aligned}
&\sup_{x \in S} \|F_n(x) - F(x)\| \\
&\quad = \sup_{x \in S} \|G_{nk} \cdot a_k(x) - G_k \cdot a_k(x)\| = \sup_{x \in Sa_k} \|G_{nk}(x) - G_k(x)\| \\
&\quad < \sup_{x \in Sa_k} \|g_{nk}(x) - f_n(x)\| + \sup_{x \in Sa_k} \|f_n(x) - f(x)\| \\
&\qquad + \sup_{x \in Sa_k} \|f(x) - g_k(x)\| \\
&\quad < \sup_{\|x\| < M} \|g_{nk}(x) - f_n(x)\| \\
&\qquad + \sup_{\|x\| < M} \|f_n(x) - f(x)\| + \sup_{\|x\| < M} \|f(x) - g_k(x)\| \\
&\quad < \|g_{nk} - f_n\| M + \|f_n - f\| M + \|f - g_k\| M < \varepsilon.
\end{aligned}$$

6. COROLLARY. *Let X be a locally compact space and $C_b(X)$ the algebra of bounded continuous real-valued functions on X. Let $A = C_0(X)$, the set of functions in $C_b(X)$ that vanish at infinity. Then $(C_b(X), \beta_A)^*$ is isomorphic to $M(X)$ where $M(X)$ is the set of bounded Radon measures on X.*

PROOF. This follows from Theorem 5 since $C_0(X)$ is a closed two-sided ideal in $C_b(X)$ and $C_0(X)^* = M(X)$ by the Riesz Representation Theorem.

This is R. C. Buck's result published in 1958 [1].

Let A be a B^*-algebra. The double centralizer on A, denoted as $M_d(A)$, is the set of pairs (T', T'') of mappings from A into A that satisfy $a(T'b) = (T''a)b$ for all $a, b \in A$. If $(T', T'') \in M_d(A)$, then T' and T'' are continuous linear maps on A and $\|T'\| = \|T''\|$, so that under the usual operations of addition and multiplication, $M_d(A)$ is a Banach algebra where $\|(T', T'')\| = \|T''\|$. By defining $(T', T'')^* = (T''^*, T'^*)$, where $T'^*(a) = (T'(a^*))^*$ and $T''^*(a) = (T''(a^*))^*$ for all $a \in A$, then $(T', T'')^* \in M_d(A)$ and thus $M_d(A)$ is a B^*-algebra.

If we define a map μ: $A \to M_d(A)$ by the formula $\mu(a) = (L_a, R_a)$ where $L_a(x) = ax$ and $R_a(x) = xa$ for all $x \in A$, then μ is an isometric *-

isomorphism from A into $M_d(A)$ and $\mu(A)$ is a closed two-sided ideal in $M_d(A)$ [2]. If A is commutative then $M_d(A)$ is isometrically $*$-isomorphic to the algebra of multipliers as studied by Wang [6]. For a more detailed account of the theory of double centralizers on a B^*-algebra, we refer the reader to [2].

7. COROLLARY. *If A is a B^*-algebra then $(M_d(A), \beta_A)^*$ under the strong topology is a Banach space that is isometrically isomorphic to A^*.*

PROOF. The isomorphism follows from Theorem 5, since A can be considered as a closed two-sided ideal of $M_d(A)$. Since a B^*-algebra has an approximate identity uniformly bounded by one, it follows from Lemma 4 that the β_A bounded subsets of $M_d(A)$ are norm bounded. Hence the strong topology on $(M_d(A), \beta_A)^*$ is the usual norm topology. The isometry follows from the fact that $f(X) = \lim_\alpha f(X\mu(e_\alpha))$ where $f \in (M_d(A), \beta_A)^*$, $X \in M_d(A)$ and $\{e_\alpha\}$ is an approximate identity on A.

This is D. C. Taylor's result published in 1970 [5].

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