THE STRICT DUAL OF $B^*$-ALGEBRAS

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Abstract. If $A$ is a closed two-sided ideal in the $B^*$-algebra $X$, then $(X, \beta_A)^*$ with the strong topology is isomorphic to $A^*$, where $\beta_A$ is the strict topology on $X$.

Introduction. Let $C_b(X)$ be the Banach space of all real-valued bounded continuous functions on $X$. It is well known that if $X$ is compact then $(C_b(X), \sigma)^*$ is isomorphic to $M(X)$ where $M(X)$ is the bounded Radon measure on $X$ and $\sigma$ is the usual topology of uniform convergence. However, if $X$ is only locally compact Hausdorff then $(C_b(X), o)^*$ is isomorphic to $M_\beta(X)$ where $\beta X$ is the Stone-Čech compactification of $X$. In 1958, R. C. Buck [1] showed that if $C_b(X)$ is given the strict topology generated by $C_0(X)$, denoted as the $\beta$ topology, then $(C_b(X), \beta)^*$ is isomorphic to $M(X)$. We show that this result is essentially due to the fact that $C_0(X)$ is a closed two-sided ideal of $C_b(X)$.

1. Definition. Let $X$ be a $\beta^*$-algebra and $A$ a closed two-sided ideal in $X$. The strict topology of $X$ with respect to $A$, denoted as the $\beta_A$ topology on $X$, is the locally convex topology generated by the seminorms $(\lambda_a)_{a \in A}$ and $(\rho_a)_{a \in A}$ where $\lambda_a(x) = ||ax||$ and $\rho_a(x) = ||xa||$. We will denote $X$ with the $\beta_A$ topology by the pair $(X, \beta_A)$. It is clear that $\beta_A$ is a vector space topology in which multiplication is separately continuous.

2. Proposition. Let $A$ be a closed two-sided ideal in the $B^*$-algebra $X$. Then $A$ is $\beta_A$ dense in $X$.

Proof. C. E. Rickart [3] has shown that $A$ is an invariant subspace with respect to involution and, hence, $A$ can be considered as a $\beta^*$-algebra. If $\{e_a|a \in A\}$ is an approximate identity in $A$ [3, p. 245], then $e_\lambda x + xe_\lambda - e_\lambda xe_\lambda$ converges to $x$ in the $\beta_A$ topology for each $x \in X$. Since $A$ is a two-sided ideal, the set $\{e_\lambda x + xe_\lambda - e_\lambda xe_\lambda|\lambda \in \Lambda\}$ is contained in $A$ and, hence, the result follows.

3. Proposition. Let $A$ be a closed two-sided ideal in the $B^*$-algebra $X$. Then $(X, \beta_A)^* = \{f \cdot a|a \in A, f \in X^*\}$. 

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Proof. Let \( f \in (X, \beta_A)^* \) and \( \phi \) be the inclusion mapping of \( A \) into \( X \). Then \( f \phi \in A^* \). Since \( A \) is a B*-algebra there exists an \( a \in A \) and \( g \in A^* \) such that \( f \phi = g \cdot a \) where \( g \cdot a(x) = g(ax) \) [5]. By the Hahn-Banach theorem \( g \) can be extended to an \( h \in X^* \) such that \( g = h \phi \). Let \( \{e_a\} \) be an approximate identity for \( A \). Since \( \{e_a x + xe_a - e_a xe_a\} \) converges to \( x \) in the \( \beta_A \) topology and \( A \) is a closed two-sided ideal, we have that

\[
f(x) = \lim_{a} f(e_a x + xe_a - e_a xe_a) = \lim_{a} g \circ a(e_a x + xe_a - e_a xe_a) = g(ax) = h(ax) = h \cdot a(x).
\]

To get the reverse inclusion, it is sufficient to observe that if \( x_a \) converges to \( x \) in the \( \beta_A \) topology and \( a \in A \), then \( ax_a \) converges to \( ax \) in the norm topology. Thus, for \( h \in X^* \), \( h \cdot a(x_a) \) converges to \( h \cdot a(x) \) and, therefore, \( h \cdot a \) is \( \beta_A \) continuous.

4. Lemma. Let \( A \) be a closed two-sided ideal in the B*-algebra \( X \) and \( S \) a \( \beta_A \) bounded subset of \( X \). Then \( \sup\{||x_a|| : x \in S, a \in A, ||a|| < 1\} \) is finite.

Proof. For each \( x \in S \), define \( T_x \) from \( A \) into \( A \) by \( T_x a = xa \). The map \( T_x \) is clearly linear and continuous. Since \( S \) is a \( \beta_A \) bounded subset, for each \( a \in A \), there exists an \( M(a) > 0 \) such that \( \sup_{x \in S} \{||T_x a|| = ||xa||\} < M(a) \). Hence, by the uniform boundedness principle, there exists an \( M > 0 \) such that \( ||T_x|| < M \) for all \( x \in S \). But

\[
||T_x|| = \sup_{||a|| < 1} ||T_x a|| = \sup_{||a|| < 1} \{||x a|| : a \in A, ||a|| < 1\}
\]

and, hence, the Lemma follows.

The strong topology on \((X, \beta_A)^*\) is defined to be the topology of uniform convergence on the \( \beta_A \) bounded subsets of \((X, \beta_A)\). Using this topology, we have the following result.

5. Theorem. If \( A \) is a closed two-sided ideal in the B*-algebra \( X \), then \(((X, \beta_A)^*, \gamma)\) is isomorphic to \( A^* \), where \( \gamma \) is the strong topology on \((X, \beta_A)^*\).

Proof. From Proposition 2 we have that \( A \) is \( \beta_A \) dense in \( X \). Let \( \phi \) be the inclusion mapping of \( A \) into \( X \) and let \( \phi^* \) be the adjoint mapping of \((X, \beta_A)^*\) into \( A^* \). We now show that \( \phi^* \) is an isomorphism.

Since \( A \) is \( \beta_A \) dense in \( X \), it follows that \( \phi^* \) is one-to-one. Let \( f \in A^* \). Then \( f = g \cdot a \) for some \( a \in A \) and \( g \in A^* \) where \( g \cdot a(x) = g(ax) \) [5]. By the Hahn-Banach theorem, we can extend \( g \) to all of \( X \) to obtain \( \tilde{g} \in X^* \) such that \( \tilde{g}|_A = g \). Since \( \phi^*(\tilde{g} \cdot a)(b) = \tilde{g} \cdot a(\phi(b)) = g \cdot a(b) \) for all \( b \in A \), it follows that \( \phi^* \) is onto.

The continuity of \( \phi^* \) is true since norm bounded sets are also \( \beta \) bounded sets. To show that \( \phi^{*-1} \) is continuous, it is sufficient to show that if \( \{f_n\} \) is a sequence in \( A^* \) that converges to \( f \in A^* \), then \( \phi^{*-1}(f_n) = F_n \) converges in the strong topology to \( \phi^{*-1}(f) = F \).
Let $S$ be a $\beta_A$ bounded set. By Lemma 4, there exists a number $M > 0$ such that
\[ \sup \{ \| xa \| : x \in S, a \in A \text{ and } \| a \| < 1 \} < M. \]

Let $\varepsilon > 0$. Since $\{ f_n \}$ converges to $f$ there exists an $N$ such that for all $n > N$, $\| f_n - f \| < \varepsilon / 3M$. Since $\{ f_n \} \in Z^*$ and $\{ f \}$ is totally bounded, for each $k > 0$, there exists $a_k \in A$, $g_k \in A^*$ and $g_{nk} \in A^*$ satisfying
\begin{align*}
(5.1) & \quad f_n = g_{nk} \cdot a_k \text{ and } f = g_k \cdot a_k, \\
(5.2) & \quad \| f_n - g_{nk} \| < 1/k \text{ and } \| f - g_k \| < 1/k, \\
(5.3) & \quad \| a_k \| < 1 [4].
\end{align*}

Now $F_n = G_{nk} \cdot a_k$ and $F = G_k \cdot a_k$, where $G_{nk}$ is a Hahn-Banach extension of $g_{nk}$ and $G_k$ is a Hahn-Banach extension of $g_k$. We choose $k$ such that $1/k < \varepsilon / 3M$. The continuity of $\phi^{*-1}$ now follows from
\begin{align*}
\sup_{x \in S} || F_n(x) - F(x) || \\
& = \sup_{x \in S} \| G_{nk} \cdot a_k(x) - G_k \cdot a_k(x) \| = \sup_{x \in S} \| G_{nk}(x) - G_k(x) \| \\
& < \sup_{x \in S} \| g_{nk}(x) - f_n(x) \| + \sup_{x \in S} \| f_n(x) - f(x) \| \\
& + \sup_{\| x \| < M} \| f(x) - g_k(x) \| \\
& < \| g_{nk} - f_n \| M + \| f_n - f \| M + \| f - g_k \| M < \varepsilon.
\end{align*}

6. Corollary. Let $X$ be a locally compact space and $C_b(X)$ the algebra of bounded continuous real-valued functions on $X$. Let $A = C_0(X)$, the set of functions in $C_b(X)$ that vanish at infinity. Then $(C_b(X), \beta_A)^*$ is isomorphic to $M(X)$ where $M(X)$ is the set of bounded Radon measures on $X$.

Proof. This follows from Theorem 5 since $C_0(X)$ is a closed two-sided ideal in $C_b(X)$ and $C_0(X)^* = M(X)$ by the Riesz Representation Theorem.

This is R. C. Buck's result published in 1958 [1].

Let $A$ be a $B^*$-algebra. The double centralizer on $A$, denoted as $M_d(A)$, is the set of pairs $(T', T'')$ of mappings from $A$ into $A$ that satisfy $a(T'b) = (T''a)b$ for all $a, b \in A$. If $(T', T'') \in M_d(A)$, then $T'$ and $T''$ are continuous linear maps on $A$ and $\| T' \| = \| T'' \|$, so that under the usual operations of addition and multiplication, $M_d(A)$ is a Banach algebra where $\| (T', T'') \| = \| T'' \|$. By defining $(T', T'')^* = (T''^*, T'^*)$, where $T'^*(a) = (T'(a^*))^*$ and $T''^*(a) = (T''(a^*))^*$ for all $a \in A$, then $(T', T'')^* \in M_d(A)$ and thus $M_d(A)$ is a $B^*$-algebra.

If we define a map $\mu: A \to M_d(A)$ by the formula $\mu(a) = (L_a, R_a)$ where $L_a(x) = ax$ and $R_a(x) = xa$ for all $x \in A$, then $\mu$ is an isometric *-
isomorphism from $A$ into $M_d(A)$ and $\mu(A)$ is a closed two-sided ideal in $M_d(A)$ [2]. If $A$ is commutative then $M_d(A)$ is isometrically *-isomorphic to the algebra of multipliers as studied by Wang [6]. For a more detailed account of the theory of double centralizers on a $B^*$-algebra, we refer the reader to [2].

7. **Corollary.** If $A$ is a $B^*$-algebra then $(M_d(A), \beta_A)^*$ under the strong topology is a Banach space that is isometrically isomorphic to $A^*$.

**Proof.** The isomorphism follows from Theorem 5, since $A$ can be considered as a closed two-sided ideal of $M_d(A)$. Since a $B^*$-algebra has an approximate identity uniformly bounded by one, it follows from Lemma 4 that the $\beta_A$ bounded subsets of $M_d(A)$ are norm bounded. Hence the strong topology on $(M_d(A), \beta_A)^*$ is the usual norm topology. The isometry follows from the fact that $f(X) = \lim_a f(X\mu(e_a))$ where $f \in (M_d(A), \beta_A)^*$, $X \in M_d(A)$ and $\{e_a\}$ is an approximate identity on $A$.

This is D. C. Taylor's result published in 1970 [5].

**References**


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