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ON π'-CLOSURE OF π-HOMOGENEOUS GROUPS

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ABSTRACT. Let π be a set of odd primes. It is known that π'-closed groups are π-homogeneous, but that the converse does not hold in general. In this paper we prove that a finite group G which is D_π and π-homogeneous is π'-closed.

Let G be a finite group, and let $\pi(G)$ denote the set of prime divisors of $|G|$. Let π be a subset of $\pi(G)$; then G is π-homogeneous if $N_G(H)/C_G(H)$ is a π-group whenever H is a nontrivial π-group. Let $\pi' = \pi(G) - \pi$; then G is π'-closed if the set of π'-elements forms a subgroup of G.

It is known that π'-closed groups are π-homogeneous. The converse does not hold in general. For example, A_5 is $5'$-homogeneous, but A_5 is not 5-closed. Thus, some extra conditions are necessary in order that π-homogeneity implies π'-closure. We shall say that G is a D_π group if all maximal π-subgroups of G are conjugate S_π-subgroups. It should be noted that we do not assume all S_π-subgroups of G are solvable. Z. Arad [1] has conjectured that if π is a set of primes, then D_π and π-homogeneity imply π'-closure. This conjecture is proved in the theorem below.

THEOREM. *Let G be a finite group such that G is D_π and π-homogeneous, then G is π'-closed.*

Throughout this paper we assume $G \in$ Hypothesis A.

HYPOTHESIS A. Let π be a set of primes. G is a D_π π-homogeneous group of minimal order such that G is not π'-closed.

We will argue by contradiction to show that no such group G ($G \in$ Hypothesis A) exists.

If $G \in$ Hypothesis A, clearly $\pi(G) \neq \pi$. Let D denote a maximal π-subgroup of G. If H is a subgroup of G, and τ is a set of primes, then let $|H|_\tau$ be defined by $|H| = |H|_\tau s$ where $(s, \tau) = 1$.

The following two results were proved in [2].

LEMMA 1 [2, LEMMA 2.3]. *Subgroups and epimorphic images of π-homogeneous groups are π-homogeneous.*

LEMMA 2 [2, LEMMA 2.4]. *If K is a normal subgroup of the π-homogeneous group H, and if K and H/K are π'-closed, then H is π'-closed.*

LEMMA 3. *Assume $G \in$ Hypothesis* A; *then* $|O_\pi(G)| = 1$.

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PROOF. We will assume $|O_\pi(G)| > 1$ and obtain a contradiction.

Let $\overline{H}$ denote the image of a subset $H \subseteq G$ in $G/O_\pi(G)$. Since G is D_π, $O_\pi(G) \subseteq D^g$ for all $g \in G$. Order arguments imply that the subgroups $\overline{D^g}$ are S_π subgroups of $\overline{G}$.

Let $\overline{H}$ be a maximal π-subgroup of $\overline{G}$. The fundamental homomorphism theorem implies $|H| \mid |G|_\pi$. Now G a D_π-group implies $H \subseteq D^g$ for $g \in G$. Thus $\overline{H} \subseteq \overline{D^g}$. The maximality of $\overline{H}$ implies $\overline{H} = \overline{D^g}$. Hence $\overline{G}$ is D_π.

Lemma 1 implies $\overline{G}$ is π-homogeneous. Now $|O_\pi(G)| > 1$ implies $|\overline{G}| < |G|$. Hence $G \in$ Hypothesis A implies $\overline{G}$ is π'-closed. Now $O_\pi(G)$ is trivially π'-closed so that Lemma 2 implies G is π'-closed. Thus if $G \in$ Hypothesis A, then $O_\pi(G) = 1$.

LEMMA 4. *Assume $G \in$ Hypothesis* A *and S is a nontrivial p-subgroup of D. Then the following conditions hold*:

(i) *$N_G(S)$ is π'-closed*;

(ii) $N_G(S) = N_D(S)O_{\pi'}(C_G(S))$;

(iii) *$S \subseteq D^w$ implies $D^w = D^y$ where $y \in O_{\pi'}(C_G(S))$.*

PROOF. Lemma 1 implies $N_G(S)$ is π-homogeneous and Lemma 3 implies $|N_G(S)| < |G|$. Hence, in order to show that $N_G(S)$ is π'-closed it is sufficient to show that $N_G(S)$ is D_π.

Suppose K is a maximal π subgroup of $N_G(S)$. Clearly $K \supseteq S$. Since G is D_π, $K \subseteq D^g$ for some $g \in G$. Hence $K = N_{D^g}(S)$ where $S \subseteq D^g$.

Let $p^n = |D|_p$. We will proceed by induction on $p^n/|S|$. If $p^n/|S| = 1$, then S is a Sylow p-subgroup of D. Suppose that $S \subseteq D^w \cap D$. Sylow's theorems imply $S = S^{dw}$ where $d \in D$. Hence

$$N_{D^w}(S) = N_{D^w}(S^{dw}) = N_{D^{dw}}(S^{dw}) = (N_D(S))^{dw}.$$

Now $dw \in N_G(S)$ implies

$$(*) \qquad N_{D^w}(S) = (N_D(S))^r \quad \text{where } r \in N_G(S).$$

Now let K be a maximal π-subgroup of $N_G(S)$; then $K = N_{D^w}(S)$ where $S \subseteq D^w$. Thus ($*$) implies K is conjugate in $N_G(S)$ to $N_D(S)$. In particular, $N_D(S)$ is a maximal π-subgroup of $N_G(S)$. Moreover, all maximal π-subgroups are conjugate in $N_G(S)$ to $N_D(S)$. It follows easily that $N_G(S)$ is D_π. Hence, $N_G(S)$ is π'-closed. Since $N_G(S)$ is π-homogeneous, we see that $O_{\pi'}(N_G(S)) = O_{\pi'}(C_G(S))$. Order arguments now imply $N_G(S) = N_D(S)O_{\pi'}(C_G(S))$.

If $S \subseteq D^w$, then $S = S^{dw}$ for $d \in D$ implies $dw \in N_G(S)$. Thus, $dw = d_2 y$ where $d_2 \in N_D(S)$ and $y \in O_{\pi'}(C_G(S))$. Hence $D^w = D^y$.

Now suppose $p^n > p^n/|S| > 1$ and the lemma is true for all T such that $p^n/|T| < p^n/|S|$ (i.e., $|T| > |S|$).

Let S_1 be a Sylow p-subgroup of $N_G(S)$, and let K be a maximal π-subgroup of $N_G(S)$ such that $S_1 \subseteq K$. Then $K = N_{D^f}(S)$ where $S \subseteq D^f$. Now S is not a Sylow p-subgroup of D. Hence Sylow's theorems imply if T is a Sylow p-subgroup of $N_D(S)$, then $|T| > |S|$. Moreover, $T \subseteq S_1^r$ where $r \in$

$N_G(S)$. Thus $T \subseteq K^r \cap D \subseteq D^{fr} \cap D$. Since $|T| > |S|$, $D^{fr} = D^y$ where $y \in O_{\pi'}(C_G(T))$. Hence, $D^f = D^{yr^{-1}}$, which implies

$$N_{D^f}(S) = N_{D^{yr^{-1}}}(S) = \left(N_D(S^{y^{-1}})\right)^{yr^{-1}}.$$

However, $S \subseteq T$ implies $y \in C_G(S)$. Hence, $N_{D^f}(S) = (N_D(S))^{yr^{-1}}$ where $yr^{-1} \in N_G(S)$. Thus $N_{D^f}(S)$ is conjugate to $N_D(S)$ in $N_G(S)$. In particular, $N_D(S)$ is a maximal π-subgroup of $N_G(S)$ and T is a Sylow p-subgroup of $N_G(S)$.

Suppose $S \subseteq D^w$; then S is not a Sylow p-subgroup of D^w. Hence if V is a Sylow p-subgroup of $N_{D^w}(S)$, then $|V/S| \geqslant p$. Sylow's theorems imply $V \subseteq T^r$ where $r \in N_G(S)$. Thus $V = T_1^r$ where $T_1 \subseteq T$ and $T_1 \supset S$. Hence $T_1 = V^{r^{-1}} \subseteq D^{wr^{-1}} \cap D$. Now $|T_1| > |S|$ implies $D^{wr^{-1}} = D^x$ where $x \in O_{\pi'}(C_G(T_1))$. Hence $x \in C_G(S)$, so that $D^w = D^{xr}$, where $xr \in N_G(S)$. Thus

$$N_{D^w}(S) = N_{D^{xr}}(S) = \left(N_D(S^{r^{-1}x^{-1}})\right)^{xr} = (N_D(S))^{xr}.$$

Thus $N_{D^w}(S)$ is conjugate in $N_G(S)$ to $N_D(S)$. In particular, $N_{D^w}(S)$ is a maximal π-subgroup of $N_G(S)$. If K is any maximal π-subgroup of $N_G(S)$, then $K = N_{D^w}(S)$ for some $D^w \supseteq S$. Hence $K = (N_D(S))^z$, where $z \in N_G(S)$. It follows that $N_G(S)$ is D_π. Lemmas 1 and 3 imply $N_G(S)$ is π'-closed. Now $O_{\pi'}(N_G(S)) = O_{\pi'}(C_G(S))$, and order arguments imply $N_G(S) = N_D(S)O_{\pi'}(C_G(S))$.

If $S \subseteq D^w$, then the previous paragraph implies $N_{D^w}(S)$ is a maximal π-subgroup of $N_G(S)$. Since $N_G(S) = N_D(S)O_{\pi'}(C_G(S))$, $N_{D^w}(S) = (N_D(S))^x$ for $x \in O_{\pi'}(C_G(S))$. Thus $D^{wx^{-1}} \cap D \supseteq T$. Now $|T| > |S|$ implies $D^{wx^{-1}} = D^u$ where $u \in O_{\pi'}(C_G(T))$. Thus $D^w = D^y$ where $y = ux$ clearly is contained in $O_{\pi'}(C_G(S))$. The lemma follows by induction.

LEMMA 5. *Assume $G \in$ Hypothesis* A. *Let $z \in D^{\#}$; then $C_G(z)$ is π'-closed. Moreover, $z \in D^w$ implies $D^w = D^y$ where $y \in O_{\pi'}(C_G(z))$.*

PROOF. Let z be an element of minimal order such that the lemma fails. Suppose $|\langle z\rangle| = p^k$ where p is a prime. Lemma 4 implies a contradiction. Thus we may assume $z = z_1z_2$ where $|\langle z_1\rangle| = p^n > 1$, $(|\langle z_2\rangle|, p) = 1$, and $z_2 \neq 1$. Clearly $C_G(z) = C_G(z_1) \cap C_G(z_2)$. Since $1 < |\langle z_i\rangle| < |\langle z\rangle|$ for $i = 1, 2$, $C_G(z_i)$ is π'-closed for $i = 1, 2$. Hence $C_G(z)$ is π'-closed.

Now $z \in D \cap D^w$ implies $z_i \in D \cap D^w$ for $i = 1, 2$. Lemma 4 implies $D^w = D^y$ where $y \in O_{\pi'}(C_G(z_1))$. Hence, $z_2 \in D^y = D^w$ which implies $z_2^{y^{-1}} \in D$. Thus

$$\left[z_2, y^{-1}\right] = z_2^{-1}z_2^{y^{-1}} \in D \cap O_{\pi'}(C_G(z_1)) = 1.$$

Therefore,

$$y \in O_{\pi'}(C_G(z_1)) \cap O_{\pi'}(C_G(z_2)) = O_{\pi'}(C_G(z)).$$

LEMMA 6. *Assume $G \in$ Hypothesis* A. *Any two elements of $D^{\#}$ which are conjugate in G are also conjugate in D.*

PROOF. Suppose $z_1 = z_2^w$ where $z_1, z_2 \in D^{\#}$. Then $z_1 \in D \cap D^w$. Lemma 5 implies $D^w = D^y$ where $y \in O_{\pi'}(C_G(z_1))$. Hence $wy^{-1} \in N_G(D)$. Since D is a Hall π-subgroup of G, $|N_G(D)/D| \mid |G|_{\pi'}$. Theorem 6.2.1 [5] implies $N_G(D) = MD$ where M is a π'-subgroup of $N_G(D)$. Since $N_G(D)$ is π-homogeneous we see that $M \subseteq C_G(D)$. Hence, $N_G(D) = M \times D$. Now $wy^{-1} \in N_G(D)$ implies $w = duy^{+1}$ where $u \in M$. Thus $z_1 = z_2^w$ implies $z_2 = z_1^{y^{-1}u^{-1}d^{-1}} = z_1^{d^{-1}}$. Thus z_1 is conjugate to z_2 in D.

PROOF OF THEOREM. In order to prove the Theorem, we will show that G satisfies the hypothesis of Theorem 8.22 of [6] with $H = D$. Let E be an elementary subgroup of G such that $|E| \mid |D|$. Since G is D_π, $E \subseteq D^g$ for some $g \in G$. Hence E is conjugate to a subgroup of D. Lemma 6 implies whenever two elements of D are conjugate in G, then they are conjugate in D. Theorem 8.22 [6] now implies G is π'-closed.

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