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A NONSPECTRAL BIRKHOFF-REGULAR DIFFERENTIAL OPERATOR

PHILIP W. WALKER

ABSTRACT. It is shown by example that Birkhoff-regular differential operators need not be spectral.

In his well-known paper [2, p. 222] Dunford stated that Birkhoff-regular operators are spectral. While many are spectral [3, XIX.4.16] we have found that some are not.

Our example is the operator L induced in $L^2(0,1)$ by the boundary value problem

$$-y'' = \lambda y \quad \text{on } [0,1], \tag{1}$$

$$y'(0) - y'(1) - y(1) = 0, \quad \text{and} \quad y(0) = 0. \tag{2}$$

L is given by $Ly = -y''$ for all y in $L^2(0,1)$ such that y' is absolutely continuous on $[0,1]$, (2) holds and $y'' \in L^2(0,1)$.

It may be verified that (1)–(2) is regular in the sense of Birkhoff ([5] or [4, §4.8]), and following [1, §§7.2 and 12.2] or [4, §3.7] it may be verified that, for each positive integer k, $(2k\pi)^2$ is an eigenvalue. (There are other eigenvalues but these are not important for our purpose here.)

By computing the Green's function and calculating its residue at $(2k\pi)^2$ [4, §3.8], it may be concluded that, for each positive integer k, the projection $E((2k\pi)^2; L)$, which we abbreviate by E_k, is given by

$$(E_k f)(t) = -\int_0^1 (\sin 2k\pi t)(4k\pi \cos 2k\pi s - 2\sin 2k\pi s) f(s)\, ds$$

for each $f \in L^2(0,1)$.

Let $f_k(s) = \cos 2k\pi s$ for each positive integer k and each $s \in [0,1]$. Then $\|f_k\| = \sqrt{2}/2$ and $\|E_k f_k\| = \sqrt{2}\, k\pi$ for each positive integer k ($\|\cdot\|$ is the usual $L^2(0,1)$ norm). Thus the sequence of projections $\{E_k\}$ is not bounded, so by [3, XVIII.2.33], L is not spectral.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004