CANONICAL OBJECTS IN KIRILLOV THEORY
ON NILPOTENT LIE GROUPS

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ABSTRACT. It is shown that to each element $f$ in the dual space of the Lie
algebra of a nilpotent Lie group there is a uniquely defined subgroup $K_f$ for
which the representation corresponding to $f$ is inducible from a square-
integrable-modulo-its-kernel representation of $K_f$.

I. Introduction. Let $G$ be a connected, simply connected nilpotent Lie group
with Lie algebra $\mathfrak{g}$. Let $U$ be an irreducible unitary representation of $G$.
Under the Kirillov correspondence $U$ corresponds to a unique orbit $\mathcal{O}$ of the
coa-adjoint representation $\text{ad}^*$ of $G$ in the dual space $\mathfrak{g}^*$. The correspondence
is obtained by selecting an $f$ in $\mathfrak{g}$ and a polarization for $f$ and then forming
an appropriate induced representation (see [1]). The polarizations correspond-
ing to a given $f$ are highly nonunique and noncanonical. In this paper we ask
the following question: To what extent is it possible to describe $U$ by means
of objects canonically defined by $\mathcal{O}$ (or $f$)? Our main result is the following
theorem.

Theorem. Associated with each $f \in \mathfrak{g}$ there is a canonical subalgebra $\mathfrak{k}_f$ with
the following properties:

(a) If $K_f$ is the corresponding subgroup to $\mathfrak{k}_f$ and $U_f^\infty$ is the representation of
$K_f$ corresponding to $f|\mathfrak{k}_f$ then $\text{ind}(K_f, G, U_f^\infty)$ (the representation induced by
$U_f^\infty$) is irreducible and equivalent to $U$.

(b) $K_f$ is invariant under any automorphism which fixes $f$.

(c) $U_f^\infty$ is square integrable modulo its kernel (see [4]).

This theorem seems useful from several points of view. $\mathfrak{k}_f$ of course must
contain polarizations for $f$. Thus there is a distinguished class of polarizations.
Furthermore, let $\mathfrak{k}_f^\infty$ be the radical of $f|\mathfrak{k}_f$ i.e.

$$\mathfrak{k}_f^\infty = \{X \in \mathfrak{k}_f \mid f([X, \mathfrak{k}_f^\infty]) = 0\}.$$

Then $f([\mathfrak{k}_f^\infty, \mathfrak{k}_f^\infty]) = 0$. Let $H_f^\infty$ be the corresponding subgroup. Then $\exp(f \circ \log) | H_f^\infty = \chi_f^\infty$ defines a character of $H_f^\infty$. Since $U_f^\infty$ is square integrable
modulo its kernel, $\text{ind}(H_f^\infty, K_f, \chi_f^\infty)$ is primary and quasi-equivalent to $U_f^\infty$
(see [4]). Hence $\text{ind}(H_f^\infty, G, \chi_f^\infty)$ is primary and quasi-equivalent to $U$. The
subgroups $H_f^\infty$ can be used to describe in a canonical fashion the primary

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projections onto the primary subspaces of nilmanifolds in much the same way that polarizations are used in the character formulas of [2]. We shall go into this in a later paper.

II. Proofs. In this section we shall define and study $K_\infty$. $K_\infty$ is defined by means of $\mathcal{K}_\infty$. We shall at first assume only that $G$ is a connected, simply connected, solvable Lie group. This will complicate some of our proofs but we feel that it sheds more light on the subject.

Let $f \in \mathfrak{g}^*$ and let $\mathcal{O}(f)$ be its orbit under $\text{ad}^*$. Let

$$\mathcal{S}(f) = \{ \lambda \in \mathfrak{g}^* | \mathcal{O}(f) + t\lambda = \mathcal{O}(f) \text{ for all } t \in \mathbb{R} \}. $$

$\mathcal{S}(f)$ is a subspace of $\mathfrak{g}^*$ which is invariant under $\text{ad}^*$. Let $\mathcal{K}(f) = \bigcap \ker \lambda$ ($\lambda \in \mathcal{S}(f)$).

**Lemma 1.** $\mathcal{K}(f)$ is an ideal in $\mathfrak{g}$.

**Proof.** $\mathcal{K}(f)$ is invariant under $\text{ad}(\exp tX)$ for all $X \in \mathfrak{g}$. Differentiating we see $[X, \mathcal{K}(f)] \subset \mathcal{K}(f)$. Q.E.D.

Now we define a sequence of subalgebras of $\mathfrak{g}$ as follows:

$$\mathcal{K}_1(f) = \mathcal{K}(f), \quad \mathcal{K}_k(f) = \mathcal{K}(f|\mathcal{K}_{k-1}(f)).$$

Let $\mathcal{K}_\infty(f) = \bigcap \mathcal{K}_k(f)$ ($k \in \mathbb{N}$).

**Lemma 2.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. Let $A: \mathfrak{g}_1 \to \mathfrak{g}_2$ be an automorphism and let $f_2 \in \mathfrak{g}_2^*$ and $f_1 \in \mathfrak{g}_1^*$ be such that $f_2 \circ A = f_1$. Then $\mathcal{K}_\infty(f_2) = A(\mathcal{K}_\infty(f_1))$.

**Proof.** Clearly $A^*(\mathcal{S}(f_2)) = \mathcal{S}(f_1)$. Hence $A(\mathcal{K}_1(f_1)) = \mathcal{K}_1(f_2)$. The result follows by induction since $\mathcal{K}_\infty(f) = \mathcal{K}_\infty(f|\mathcal{K}_1(f))$. Q.E.D.

We shall require a criterion for deciding when $\mathcal{K}_\infty = \mathfrak{g}$. First we need some notation. If $f \in \mathfrak{g}^*$ and $\mathfrak{m}$ is subspace of $\mathfrak{g}$, let

$$\mathfrak{m}^f = \{ X \in \mathfrak{g} | f([X, M]) = 0 \text{ for all } M \in \mathfrak{m} \}. $$

Let $\mathfrak{m} = \mathfrak{m}^f$. Let $R = \{ x \in G | \text{ad} xf = f \}$.

**Theorem 1.** $\mathcal{K}_\infty \neq \mathfrak{g}$ iff there is a proper ideal $\mathfrak{g}_1$ of $\mathfrak{g}$ containing $\mathfrak{m}$.

**Proof.** $\mathcal{K}_\infty \neq \mathfrak{g}$ iff $\mathcal{K}_1 \neq \mathfrak{g}$, so it suffices to consider $\mathcal{K}_1$. We claim that $\mathcal{K}_1$ contains $\mathfrak{m}$. Let

$$K = \{ x \in G | \text{ad}^* x(f)|\mathcal{K}_1 = f|\mathcal{K}_1 \}. $$

Since $\mathcal{K}_1^\perp$ (the annihilator of $\mathcal{K}_1$) is $\mathfrak{s}$, we have $f + \mathcal{K}_1^\perp \subset \mathcal{O}(f)$. It follows that $\text{ad}^* K(f) = f + \mathcal{K}_1^\perp$. Hence $\dim R \setminus K = \dim \mathcal{K}_1^\perp$. The Lie algebra $\mathcal{K}$ of $K$ is $\mathcal{K}_1^\perp$. Then $\dim \mathcal{K} - \dim \mathfrak{m} = \dim R \setminus K = \dim \mathcal{K}_1^\perp$. The kernel of the map $X \to f([X, f])$ of $\mathcal{K}$ into $\mathfrak{m}^*$ is $\mathfrak{m}$ and hence the image of $\mathcal{K}$ has $\dim \mathcal{K} - \dim \mathfrak{m} = \dim \mathcal{K}_1^\perp$. The image is contained in $\mathcal{K}_1^\perp$ so
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\[ \mathfrak{K}_1 = \{ f([X, \cdot]) | X \in \mathfrak{K} \}. \]

Hence \( \mathfrak{K}_1 = \mathfrak{K}^4 \) and thus \( \mathfrak{S} \subseteq \mathfrak{K}_1 \), proving that \( \mathfrak{S} \) is contained in an ideal.

Conversely, let \( R_0 \) be the subgroup corresponding to \( \mathfrak{S} \). If \( \mathfrak{S} \) is an ideal containing \( \mathfrak{S}_x \), we claim that \( \mathfrak{S}_x \subseteq \mathfrak{S}(f) \). Let \( \mathfrak{L} = \mathfrak{S}_x \). Let \( L \) be the corresponding analytic subgroup of \( G \). The mapping \( \phi: x \to \text{ad}^* x(f) \) of \( L \) into \( \mathfrak{S}(f) \) maps into the affine subspace \( f + \mathfrak{S}_x \) and is constant on cosets of \( R_0 \) in \( L \). (Note that \( R_0 \subseteq L \).) It gives rise to a \( C^\infty \) map \( \phi \) of \( R_0 \) into \( f + \mathfrak{S}_x \). The tangent space at \( R_0 e \) in \( R_0 \) is canonically isomorphic with \( \mathfrak{S} \) while the tangent space at \( f \) in \( f + \mathfrak{S}_x \) is \( \mathfrak{S}_x \). The differential of \( \phi \) at \( R_0 e \) is given by \( X + \mathfrak{S}_x \to f([X, \cdot]) \). Now the bilinear form \( B = f([\cdot, \cdot]) \) is nondegenerate on \( \mathfrak{S} \) and hence \( \dim \mathfrak{S} = \dim \mathfrak{S}_x \). It follows that \( \dim \mathfrak{S} = \dim \mathfrak{S}_x \) so \( \mathfrak{S}_x \) is surjective at \( R_0 e \). Hence \( \mathfrak{S}_x \) is nonsingular at \( R_0 e \) and \( \phi \) is an open map in a neighborhood of \( R_0 e \). In particular \( \text{ad}^* L(f) \) contains a neighborhood of \( f \) in \( f + \mathfrak{S}_x \). Similar comments hold for \( f \) replaced by \( \text{ad}^* g(f), g \in L \). It follows that \( \text{ad}^* L(f) \) is open in \( f + \mathfrak{S}_x \). The same is true for any \( f' \in f + \mathfrak{S}_x \) so the orbits of \( L \) in \( f + \mathfrak{S}_x \) are all open. Since different orbits are disjoint and \( f + \mathfrak{S}_x \) is a union of orbits, this contradicts the connectedness of \( f + \mathfrak{S}_x \) unless there is only one orbit. Hence \( \text{ad}^* L(f) = f + \mathfrak{S}_x \), showing that \( f + \mathfrak{S}_x \subseteq \mathfrak{S}(f) \). Similarly \( f' + \mathfrak{S}_x \subseteq \mathfrak{S}(f) \) for any \( f' \in \mathfrak{S}(f) \). It follows that \( \mathfrak{S}_x \subseteq \mathfrak{S}(f) \). Q.E.D.

Recall that a subalgebra \( \mathfrak{K} \) of \( \mathfrak{K} \) is said to be subordinate to \( f \) if \( f([\mathfrak{K}, \mathfrak{K}]) = 0 \).

**Corollary.** If \( \mathfrak{K} \) is nilpotent, \( \mathfrak{K}_\infty \) is subordinate to \( f \).

**Proof.** Let \( f' = f|\mathfrak{K}_\infty \). By definition of \( \mathfrak{K}_\infty \), \( \mathfrak{K}_\infty (f') = \mathfrak{K}_\infty \). Hence there is no proper ideal \( \mathfrak{S} \) containing \( \mathfrak{S}(f') \). Since \( \mathfrak{K}_\infty \) is nilpotent, this implies that \( \mathfrak{S}(f') = \mathfrak{K}_\infty \). Obviously \( \mathfrak{S}(f') \) is subordinate. Q.E.D.

**Definition.** Let \( \mathfrak{K}_x = (\mathfrak{K}_\infty)^f \cap \text{normalizer}(\mathfrak{K}_\infty) \). Let \( K_1 \) be the corresponding connected analytic subgroup.

**Proposition.** \( \mathfrak{K}_\infty = (\mathfrak{K}_\infty)^f \cap \text{normalizer}(\mathfrak{K}_\infty) \). Let \( K_1 \) be the corresponding subgroup. For \( k \in K_1 \), \( \text{ad}^* (k) f|\mathfrak{K}_1 = f|\mathfrak{K}_1 \). Hence \( \text{ad}^* (k)(\mathfrak{K}_\infty) = \mathfrak{K}_\infty \) by Lemma 2. (Recall \( \mathfrak{K}_\infty = \mathfrak{K}_\infty (f|\mathfrak{K}_1) \). It follows that \( \mathfrak{K}_1 \subseteq \mathfrak{K}_\infty \). Let \( \mathfrak{K}_{n-1}(f) = \mathfrak{K}_1(f|\mathfrak{K}_{n-1}) \). (Note that \( \mathfrak{K}_n(f) \subseteq \mathfrak{K}_{n-1}(f) \).) Then \( \sum_i \mathfrak{K}_i \subseteq \mathfrak{K}_\infty \) and \( \mathfrak{K}_\infty \cap \mathfrak{K}_1 = \mathfrak{K}_1 \). As was shown in the proof of the above theorem, \( \mathfrak{K}_1 f = \mathfrak{K}_1 f \). Hence \( \mathfrak{K}_1 f \cap \mathfrak{K}_{n-1} = \mathfrak{K}_1 \). It follows that \( \mathfrak{K}_1 f = \mathfrak{K}_1 f \cap \mathfrak{K}_{n-1} = \mathfrak{K}_1 \). Hence \( \mathfrak{K}_1 f \subseteq \mathfrak{K}_\infty \). Obviously \( \mathfrak{K}_\infty \subseteq \mathfrak{K}_1 f \) so we have shown \( \mathfrak{K}_1 f = \mathfrak{K}_\infty \). That \( \text{ad}^* K_1 f = f + \mathfrak{K}_1 f \) is the same as the argument that \( \text{ad}^* K_1 f = f + \mathfrak{K}_1 f \) done in the above theorem. Q.E.D.

**Corollary.** In the notation of the above proof \( \mathfrak{K}_\infty = \sum \mathfrak{K}_i \).

**Proof.** Both \( \mathfrak{K}_\infty \) and \( \sum \mathfrak{K}_i \) contain \( \mathfrak{S} \) and both have the same orthogonal complement under \( B_f \). Hence they have the same dimension. Q.E.D.
Corollary. Let $f_\infty = f|_{\mathcal{X}_\infty}$. Then the $K_\infty$ orbit of $f_\infty$ is an affine subspace of $\mathcal{X}_\infty$.

Now we once again assume that $G$ is nilpotent. Let $f_\infty = f|_{\mathcal{X}_\infty}$ and let $U^\infty$ be the irreducible representation of $K_\infty$ corresponding to $f_\infty$ under the Kirillov correspondence.

Theorem 2. $\text{ind}(K_\infty, G, U^\infty)$ is irreducible and corresponds to $0(f)$ under the Kirillov correspondence. Also $U^\infty$ is square integrable modulo its kernel.

Proof. From the above corollary $f_\infty$ has a flat orbit so it follows from the Moore-Wolf theorem [4] that $U^\infty$ is square integrable modulo its kernel.

To prove the irreducibility let $\mathcal{X}_1(f)$ and $\mathcal{X}_1(f)$ be as before. Let $H_1$ and $K_1$ be the corresponding subgroups. Let $K_{\infty,1} = K_\infty(f|_{\mathcal{X}_1})$. $K_1$ fixes $f|_{\mathcal{X}_1}$ so $K_1$ normalizes $\mathcal{X}_\infty$ and hence $K_1$ normalizes $K_{\infty,1}$. From Corollary 3 it follows that $K_\infty = K_1K_{\infty,1}$. Let $U^{1,1}$ be the representation of $K_{\infty,1}$ corresponding to $f|_{\mathcal{X}_{\infty,1}}$ and let $U^1 = \text{ind}(K_{\infty,1}, H_1, U^{1,1})$. By induction $U^1$ is irreducible and corresponds to $f_1 = f|_{\mathcal{X}_1}$. Let $L$ be the stabilizer of $U^1$ in $G$ – i.e.

$$L = \{ x \in G | U^1(x \cdot x^{-1}) \approx U^1 \}.$$

$x \in L$ if $\text{ad}^* x(f_1)$ is in the $H_1$ orbit of $f_1$ – i.e. iff $\text{ad}^* x(f_1) = \text{ad}^* y(f_1)$ for some $y \in H_1$. This is equivalent to saying $\text{ad}^* y^{-1} x(f_1) = f_1$ – i.e. $x \in K'_1 H_1$ where $K'_1 = \{ x \in G | \text{ad} x(f_1) = f_1 \}$. It follows from standard arguments that $K'_1$ is connected (see [1, I.3.3], e.g.) and that the Lie algebra of $K'_1$ is precisely $\mathcal{X}_1' = \mathcal{X}_1$. Hence $K'_1 = K_1$ and $L = K_1 H_1$. $U^1$ extends to a representation $V$ of $L$. In fact, $K_{\infty} = K_1 K_{\infty,1} \subset L$. Let $V = \text{ind}(K_{\infty}, L, U^\infty)$. Since $L = K_1 H_1$, $V|H_1 = \text{ind}(K_{\infty,1}, H_1, U^{1,1}) = U^1$. It now follows from Mackey's theorem [3] that $\text{ind}(L, G, V)$ is irreducible. From the theorem on inducing in stage this is just $\text{ind}(K_\infty, G, U^\infty)$. It is obvious that this representation corresponds to $f$ – simply pick a polarization for $f$ contained in $\mathcal{X}_\infty$. Q.E.D.

References


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