OPERATORS IN THE COMMUTANT
OF A REDUCTIVE ALGEBRA

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Abstract. Let $\mathcal{A}$ be a reductive algebra. It is shown that there is a subspace $\mathcal{M}$ that reduces $\mathcal{A}$ and such that the commutant of $\mathcal{A}|\mathcal{M}$ is selfadjoint and the commutant of $\mathcal{A}|\mathcal{M}^\perp$ consists of hyporeductive operators. It is then shown that under a variety of conditions, if an operator $T$ is in $\mathcal{A}'$, then $T^*\in\mathcal{A}'$.

In [5], we made use of a decomposition of reductive operators on Hilbert space to deduce some results concerning the selfadjointness of the commutants of such operators. In that paper we also observed that a similar decomposition of a reductive operator algebra is possible. In this paper we produce such a decomposition and use it to answer a number of questions first raised by Rosenthal [9].

By an algebra, we will mean a weakly closed subalgebra (with identity) of the algebra of all (bounded) operators on a separable Hilbert space. If $\mathcal{A}$ is an algebra and $\mathcal{M}$ a subspace of $\mathcal{H}$ then $\mathcal{A}\mathcal{M} = \{Af: A \in \mathcal{A} \text{ and } f \in \mathcal{M}\}$, and $\mathcal{M}$ is invariant for $\mathcal{A}$ if $\mathcal{A}\mathcal{M} \subseteq \mathcal{M}$. The lattice of subspaces invariant for $\mathcal{A}$ is denoted by $\text{Lat } \mathcal{A}$. We also denote by $\mathcal{A}^*$ the algebra $\{A^*: A \in \mathcal{A}\}$, and by $\mathcal{A}'$ the algebra $\{B: AB = BA \text{ for all } A \in \mathcal{A}\}$. Finally, $\mathcal{A}$ is reductive if $\mathcal{A}\mathcal{M} \in \text{Lat } \mathcal{A}$ implies $\mathcal{M}^\perp \in \text{Lat } \mathcal{A}$, or equivalently, if $\text{Lat } \mathcal{A} = \text{Lat } \mathcal{A}^*$.

The results for which we are aiming are Theorems 2 and 3, which state that for every reductive algebra $\mathcal{A}$ there is a subspace $\mathcal{M}_0 \in \text{Lat } \mathcal{A} \cap \text{Lat } \mathcal{A}'$, such that $\mathcal{A}|\mathcal{M}_0$ is selfadjoint and $\text{Lat}(\mathcal{A}|\mathcal{M}_0^\perp) \subseteq \text{Lat}(\mathcal{A}|\mathcal{M}_0^\perp)$. Before we begin, we remark that certain techniques developed by Hoover [4] will also yield Theorems 2 and 3.

Theorem 1. Let $\mathcal{A}$ be a reductive algebra and $\mathcal{M} \in \text{Lat } \mathcal{A}$. Let $X: \mathcal{M}^\perp \to \mathcal{M}$ and suppose that $T \in \mathcal{A}'$, where $T$ has the form

$$T = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$$

according to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Then

1. $T^* \in \mathcal{A}'$.
2. $(\text{ran } T)^\perp \in \text{Lat } \mathcal{A}$ and $(\mathcal{A}|(\text{ran } T)^\perp)'$ is selfadjoint.
3. $\ker T \in \text{Lat } \mathcal{A}$ and $(\mathcal{A}|\ker T)'$ is selfadjoint.

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Proof. (1) All decompositions of vectors and operators are with respect to \( \mathcal{M} \) and \( \mathcal{M}^\perp \). Let \( \mathcal{N} \) be the subspace \( \{ \langle Xf, f \rangle : f \in \mathcal{M}^\perp \} \). Since \( T \in \mathcal{A}' \) it is easy to check that \( \mathcal{N} \in \text{Lat } \mathcal{A} \), and thus the subspace \( \mathcal{N}^\perp = \{ \langle g, -X^*g \rangle : g \in \mathcal{M} \} \) also lies in \( \text{Lat } \mathcal{A} \).

If \( A \in \mathcal{A} \) we can decompose \( A \) as \( A_1 \oplus A_2 \), since \( \mathcal{A} \) is reductive. If \( g \in \mathcal{M} \) we have

\[
(A_1 \oplus A_2) \langle g, -X^*g \rangle = \langle A_1g, -A_2X^*g \rangle.
\]

Since this vector must lie in \( \mathcal{N}^\perp \) it follows that \( -X^*A_1g = -A_2X^*g \) for any \( g \in \mathcal{M} \), and hence that \( X^*A_1 = A_2X^* \), which implies that \( T^* \in \mathcal{A}' \).

(2) Since \( T \in \mathcal{A}' \), \( (\text{ran } T)^- \in \text{Lat } \mathcal{A} \). Let \( C \) be an operator on \( (\text{ran } T)^- \) that lies in the commutant of \( \mathcal{A} |(\text{ran } T)^- \); we want to show that \( C^* \in (\mathcal{A} |(\text{ran } T)^-) \). Let \( C_1 \) be the operator on \( \mathcal{K} \) defined by

\[
C_1f = \begin{cases} 
Cf, & f \in (\text{ran } T)^-; \\
0, & f \in \text{ran } T^+. 
\end{cases}
\]

Then \( C_1 \in \mathcal{A}' \) and it will suffice to show that \( C_1^* \in \mathcal{A}' \).

Let \( T_1 = C_1T \); then \( T_1 \in \mathcal{A}' \). Moreover, since \( \mathcal{M} \) is invariant under \( C_1 \), it is easy to see that \( T_1 \) has the same form as \( T \), namely

\[
T_1 = \begin{pmatrix} 0 & \ast \\
0 & 0 \end{pmatrix}.
\]

It follows by part (1) that \( T_1^* \in \mathcal{A}' \), that is, for any \( A \in \mathcal{A} \), \( T^*C_1^*A = AT^*C_1^* \). Since \( T^* \in \mathcal{A}' \), \( T^*(AC_1^* - C_1^*A) = 0 \), that is, \( \text{ran } (AC_1^* - C_1^*A) \subseteq \ker T^* \). On the other hand \( \text{ran } C_1^* \subseteq (\text{ran } T)^- \) and \( (\text{ran } T)^- \) reduces \( A \), so that

\[
\text{ran } (AC_1^* - C_1^*A) \subseteq (\text{ran } T)^- = \ker T^*.
\]

We conclude that \( AC_1^* - C_1^*A = 0 \), and \( C_1^* \in \mathcal{A}' \).

(3) Let \( U \) be the unitary operator \( \binom{0}{1} \). Since \( T^* \in \mathcal{A}' \), we have

\[
\begin{pmatrix} 0 & X^* \\
0 & 0 \end{pmatrix} = U^*T^*U \in (U^*\mathcal{A}U)'.
\]

By part (2), \( [(U^*\mathcal{A}U)|(\text{ran } U^*T^*U)^-]' \) is selfadjoint. Thus \( (\mathcal{A}|(\text{ran } T^-)' \) is selfadjoint and (3) follows because \( (\text{ran } T^-) = \ker T^* \).

Corollary. Let \( \mathcal{A} \) be reductive and \( \mathcal{M} \in \text{Lat } \mathcal{A} \). Let \( Y: \mathcal{M} \to \mathcal{M}^\perp \) and suppose that \( S \in \mathcal{A}' \), where \( S = (0 1) \). Then

(1) \( S^* \in \mathcal{A}' \).
(2) \( \ker S \in \text{Lat } \mathcal{A} \) and \( (\mathcal{A}|\ker S)' \) is selfadjoint.
(3) \( (\text{ran } S)^- \in \text{Lat } \mathcal{A} \) and \( (\mathcal{A}|\text{ran } S)' \) is selfadjoint.

Proof. Consider adjoints and apply the theorem.

Theorem 2. Let \( \mathcal{A} \) be a reductive algebra. There is a subspace \( \mathcal{M}_0 \) such that

(1) \( \mathcal{M}_0 \in \text{Lat } \mathcal{A} \);
(2) \( (\mathcal{A}|\mathcal{M}_0)' \) is selfadjoint;
(3) there is no nonzero subspace \( \mathcal{M} \subseteq \mathcal{M}_0^\perp \) with properties (1) and (2).
Moreover, this subspace $\mathcal{M}_0$ reduces $\mathcal{G}$ as well.

**Proof.** Let $\mathcal{F} = \{ \mathcal{M} \in \text{Lat} \mathcal{G} : (\mathcal{G} | \mathcal{M})' \text{ is selfadjoint} \}$. The family $\mathcal{F}$ is nonempty since it contains the zero subspace. Suppose that $\{ \mathcal{M}_\alpha : \alpha \in B \}$ is a chain in $\mathcal{F}$; in order to apply Zorn’s lemma we would like to show that $\mathcal{M} = \bigvee \{ \mathcal{M}_\alpha : \alpha \in B \}$ is also in $\mathcal{F}$, for which it suffices to show that $(\mathcal{G} | \mathcal{M})'$ is selfadjoint.

Let $T$ be an operator on $\mathcal{M}$ such that $T \in (\mathcal{G} | \mathcal{M})'$. To show that $T^* \in (\mathcal{G} | \mathcal{M})'$ we must show that for all $A \in \mathcal{G} | \mathcal{M}$ and for all $f \in \mathcal{M}$ we have $T^*Af = AT^*f$, and in fact it will be enough to show this equality for all $f \in \bigcup \{ \mathcal{M}_\alpha : \alpha \in B \}$, because this set is dense in $\mathcal{M}$. On the other hand, in this case $f \in \mathcal{M}_\beta$ for some $\beta \in B$.

Decompose $\mathcal{M}$ as $\mathcal{M}_\beta \oplus (\mathcal{M} \ominus \mathcal{M}_\beta)$; then (since $\mathcal{M}_\beta \in \text{Lat} \mathcal{G}$) if $A \in \mathcal{G} | \mathcal{M}$ we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$ \[Because f \in \mathcal{M}_\beta we have

$$T^*Af = \langle T_{11}^*A_1f, T_{12}^*A_2f \rangle \quad \text{and} \quad AT^*f = \langle A_1T_{11}^*f, A_2T_{12}^*f \rangle.$$ \[Since $T_{11} \in (\mathcal{G} | \mathcal{M}_\beta)'$ and $(\mathcal{G} | \mathcal{M}_\beta)'$ is selfadjoint, we have $T_{11}^*A_1 = A_1T_{11}^*$. Furthermore, the operator $(0 \, T_{12}^*)$ lies in $(\mathcal{G} | \mathcal{M}_\beta)'$ and by Theorem 1, so does its adjoint $(0_{T_{12}}^*)$ and it follows that $A_2T_{12}^* = T_{12}^*A_1$. Thus $T^*Af = AT^*f,$ $T^* \in (\mathcal{G} | \mathcal{M}_\beta)'$, and $\mathcal{M} \in \mathcal{F}$.

By Zorn’s lemma there exists a maximal element $\mathcal{M}_0$ of $\mathcal{F}$. $\mathcal{M}_0$ automatically satisfies requirements (1) and (2) of the theorem. Suppose there is a nonzero subspace $\mathcal{M}_1$ of $\mathcal{M}_0$ for which (1) and (2) hold. We assert that $\mathcal{M}_0 \oplus \mathcal{M}_1$ lies in $\mathcal{F}$, a fact which contradicts the maximality of $\mathcal{M}_0$. We must show that $(\mathcal{G} | \mathcal{M}_0 \oplus \mathcal{M}_1)'$ is selfadjoint. If $S$ is an operator on $\mathcal{M}_0 \oplus \mathcal{M}_1$ such that $S \in (\mathcal{G} | \mathcal{M}_0 \oplus \mathcal{M}_1)$ then we decompose $S$ as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where $S_{11} \in (\mathcal{G} | \mathcal{M}_0)'$ and $S_{22} \in (\mathcal{G} | \mathcal{M}_1)'$. By assumption $S_{11}^* \in (\mathcal{G} | \mathcal{M}_0)'$ and $S_{22}^* \in (\mathcal{G} | \mathcal{M}_1)'$. Moreover, the operator

$$\begin{pmatrix} 0 & S_{12} \\ 0 & 0 \end{pmatrix}$$

satisfies the hypotheses of Theorem 1 for the reductive algebra $(\mathcal{G} | \mathcal{M}_0 \oplus \mathcal{M}_1)$. Thus by that theorem the operator

$$\begin{pmatrix} 0 & 0 \\ S_{12}^* & 0 \end{pmatrix}$$

lies in $(\mathcal{G} | \mathcal{M}_0 \oplus \mathcal{M}_1)'$. Similarly we use the corollary to Theorem 1 to show that $(0_{S_{12}})$ lies in $(\mathcal{G} | \mathcal{M}_0 \oplus \mathcal{M}_1)'$. Since
we see that $S^* \in (\mathcal{A} | \mathcal{M}_0 \oplus \mathcal{M}_1)'$, and thus that $(\mathcal{A} | \mathcal{M}_0 \oplus \mathcal{M}_1)'$ is selfadjoint.

Finally, to show that $\mathcal{M}_0 \in \text{Lat } \mathcal{A}'$, suppose that $T \in \mathcal{A}'$ and write $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$

according to the decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_0^\perp$.

Since $\mathcal{M}_0$ reduces $\mathcal{A}$, the operator $S = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}$

also lies in $\mathcal{A}'$, and by Theorem 1, ker $S$ reduces $\mathcal{A}$ and $(\mathcal{A} | \text{ker}^{-} S)'$ is selfadjoint. However, ker$^{-} S$ is a subspace of $\mathcal{M}_0^\perp$ so by the maximality of $\mathcal{M}_0$ it must be that ker$^{-} S = \{0\}$, that is, $T_{12} = 0$. Similarly we can show that $T_{21} = 0$ and thus $\mathcal{M}_0$ reduces $T$.

We use the notation Red $\mathcal{B}$ to mean $\text{Lat } \mathcal{B} \cap \text{Lat } \mathcal{B}^*$, where $\mathcal{B}$ is any algebra.

**Theorem 3.** Let $\mathcal{A}$ be reductive and suppose that for no nonzero subspace $\mathcal{M}$ in $\text{Lat } \mathcal{A}$ is it true that $(\mathcal{A} | \mathcal{M})'$ is selfadjoint. Then $\text{Lat } \mathcal{A} \subseteq \text{Red } \mathcal{A}'$.

**Proof.** Suppose $\mathcal{M} \in \text{Lat } \mathcal{A}$ and $T \in \mathcal{A}'$. Decompose $\mathcal{H}$ as $\mathcal{M} \oplus \mathcal{M}^\perp$ and $T$ as $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$;

let $S = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}$.

By Theorem 1, $(\text{ran } S)^- \in \text{Lat } \mathcal{A}$ and $(\mathcal{A} | (\text{ran } S)^-)'$ is selfadjoint. Hence, by the hypothesis of the theorem, ran $S = \{0\}$, that is, $T_{12} = 0$. Similarly $T_{21} = 0$ and $\mathcal{M} \in \text{Red } \mathcal{A}'$.

C. K. Fong [1] has used the word hyporeductive to refer to an operator $T$ such that every hyperinvariant subspace of $T$ reduces $T$.

**Corollary.** Let $\mathcal{A}$ be as in Theorem 3. If $T \in \mathcal{A}'$ and $\mathcal{M}$ is hyperinvariant for $T$ then $\mathcal{M} \in \text{Red } \mathcal{A}'$. In particular, $T$ is hyporeductive.

It follows from all the above that if $\mathcal{A}$ is any reductive algebra and $T \in \mathcal{A}'$, then $T = T_1 \oplus T_2$ where $T_1^* \in (\mathcal{A} | \mathcal{M}_0)'$ and $T_2$ is hyporeductive. Thus if we desire to show that $T^* \in \mathcal{A}'$, it suffices to show that $T_2^* \in (\mathcal{A} | \mathcal{M}_0)'$.

In [9], P. Rosenthal introduced the following property which an operator $T$ may have in connection with reductive algebras:
(P) If $\mathfrak{A}$ is any reductive algebra such that $\mathfrak{A}'$ contains $T$, then $\mathfrak{A}'$ contains $T^*$.

Rosenthal then asked if $T$ has property (P) under each of the following conditions:

1. $T$ is polynomially compact,
2. $1 - T^*T$ is in some $C_p$ class,
3. $T^* - T$ is in some $C_p$ class,
4. $T$ is a part of a finite-multiplicity backward shift.

We will show that each of the above conditions implies (P), but we need a preliminary result (Lemma 2).

In [5], the following lemma is proved:

**Lemma 1.** Let $C$ be a nonzero compact operator, and suppose that $B$ is an operator such that every subspace that reduces both $B$ and $C$ and has dimension greater than 1 properly contains a nonzero subspace that reduces $B$ and $C$. Then $B$ and $C$ have a common reducing eigenvector.

The argument used to prove this lemma, with minor (and obvious) modifications will yield the following fact:

**Lemma 2.** Let $B$ be an operator and $C$ a nonzero compact operator. Suppose that

1. Every hyperinvariant subspace of $B$ reduces $B$ and $C$.
2. Every hyperinvariant subspace of $B$ of dimension greater than 1 properly contains a nonzero hyperinvariant subspace of $B$.

Then $B$ and $C$ have a common reducing eigenvector.

**Proof.** See [5, p. 230].

We are now ready to answer Rosenthal's questions. It should be remarked that C. K. Fong [2] has proved part (1) of the following theorem.

**Theorem 4.** An operator $T$ has property (P) under any one of the following conditions:

1. $T$ is polynomially compact,
2. $T^* - T$ is in some $C_p$ class,
3. $1 - T^*T$ is in some $C_p$ class,
4. $T$ is a part of some finite-multiplicity backward shift.

**Proof.** (4) follows from (3), because the multiplicity of the shift of which $T$ is a part is the rank of $\sqrt{1 - T^*T}$. (See [3, p. 278].) If this rank is finite then so is the rank of $1 - T^*T$.

Next we remark that each of the conditions (1), (2), (3) is inherited by direct summands, and that each condition guarantees the existence of hyperinvariant subspaces [8, Corollaries 6.13, 6.15, 6.16] and [7, Theorem 1.1].

We will prove in detail that (2) implies that (P) holds; the proofs for (1) and (3) are analogous. Let $C = T^* - T$ and suppose that $C$ is in some $C_p$ class. We also suppose that $\mathfrak{A}$ is a reductive algebra and that $T \in \mathfrak{A}'$. Let $\mathfrak{M}_0$ be
the subspace of Theorem 2; note that \( M_0 \) reduces \( \mathcal{E} \), \( T \), and \( C \), and that \((T^*|\mathcal{M}_0) \in (\mathcal{E}|\mathcal{M}_0)'\). Thus it suffices to consider the case where \( T \) is hyporeductive (by the remark following the Corollary to Theorem 3).

Since \( T \) is hyporeductive, the space \( \mathcal{M}_1 \) spanned by all the eigenvectors of \( T \) reduces \( \mathcal{E} \), \( T \), and \( C \), and \((T^*|\mathcal{M}_1) \in (\mathcal{E}|\mathcal{M}_1)'\) (the last statement follows by Lemma 5 of [1]); thus it suffices to consider the restriction of \( \mathcal{E} \), \( T \), and \( C \) to \( \mathcal{M}_1^+ \); i.e., we consider the case where \( T \) has no eigenvalues and is hyporeductive.

After these reductions suppose \( C \) is nonzero. Because \( T \) is hyporeductive, every hyperinvariant subspace of \( T \) reduces \( T \) and \( C \). Further, suppose \( \mathcal{M} \) is a hyperinvariant subspace of \( T \), of dimension greater than 1. Then \( \mathcal{M} \) reduces \( T \) and \( C \), and \((T|\mathcal{M})^* - (T|\mathcal{M}) \) lies in some \( C_p \) class. Thus there is a hyperinvariant subspace of \( T \) properly contained in \( \mathcal{M} \). It now follows from the assumption that \( C \) is nonzero, and from Lemma 2, that \( T \) has a reducing eigenvector; however, we reduced to the case where \( T \) has no eigenvectors. Thus it must be that \( C = 0 \), which means that \( T^* = T \) and \( T^* \in \mathcal{E}' \). The proof is complete.

To show that (1) implies (P) let \( p \) be a polynomial such that \( p(T) = C \) is compact and proceed as above. It is necessary to know that an algebraic hyporeductive operator is normal [1, Theorem 4].

To show that (3) implies (P) let \( C = 1 - T^*T \) and proceed as above.

We remark that the proof of Theorem 4 also establishes the following fact:

**Corollary.** If \( T \) is hyporeductive and any one of conditions (1) through (4) holds, then \( T \) is normal.

**References**


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