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THE LEFSCHETZ FIXED POINT THEOREM FOR COMPACT GROUPS

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ABSTRACT. It is shown that every compact group G is a Q-simplicial space where Q is any field of characteristic zero. As a consequence it follows that G satisfies a variation of the Lefschetz fixed point theorem.

It has been known for some time that the Lefschetz fixed point theorem applies to a few spaces other than just ANR spaces, especially if some care is taken to use coefficients in certain fields [2]. The case of all compact groups provides a broad class of spaces which may not have local connectivity of any order. It is shown that every compact group G satisfies the Lefschetz fixed point theorem when coefficients for the homology groups are taken in a field of characteristic zero.

1. **Compact groups.** Suppose that G is a compact group. By the analysis of A. Weil [4, pp. 88–90], G is an inverse limit of compact Lie groups $G = \operatorname{proj\,lim} G_\mu$ where each G_μ is locally isomorphic to a direct product of an abelian group and a semisimple group. In more detail, the connected component of the center of G_μ is an abelian group A_μ such that for a normal subgroup S_μ which is semisimple, $G_\mu = A_\mu \cdot S_\mu$ where $\mathbf{Z}_\mu = A_\mu \cap S_\mu$ is finite. Further $G_\mu \cong (A_\mu \times S_\mu)/\mathbf{Z}_\mu$. If $f_{\mu\nu}\colon G_\nu \to G_\mu$ is a homomorphism belonging to the inverse system, then $f_{\mu\nu}$ maps A_ν onto A_μ and S_ν onto S_μ. Now each S_μ has a finite cover by a simply connected group $\tilde{S}_\mu$ which is a finite product of simply connected simple groups. A homomorphism from one product of simply connected simple groups onto another is just a projection of a product space onto a factor space. In summary we have the following theorem.

1. THEOREM. *Let G be a compact group. Then G is an inverse limit of Lie groups $G = \operatorname{proj\,lim} G_\mu$ and if $f_{\mu\nu}\colon G_\nu \to G_\mu$ is a homomorphism belonging to this inverse system, then there is a commutative diagram of covering compact Lie groups*

$$\begin{array}{ccc} \tilde{G}_\nu & \xrightarrow{\ \tilde{f}_{\mu\nu}\ } & \tilde{G}_\mu \\ {\scriptstyle p_\nu}\big\downarrow & & \big\downarrow{\scriptstyle p_\mu} \\ G_\nu & \xrightarrow{\ f_{\mu\nu}\ } & G_\mu \end{array}$$

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such that the covering groups $\tilde{G}_\nu$ and $\tilde{G}_\mu$ are finite sheeted covering spaces, and such that $\tilde{f}_{\mu\nu}$ is a projection of a product space onto a factor space.

2. **The Q-simplicial condition.** Let α be a family of subsets of a space X. For a subset B of X let

$$\mathrm{St}_\alpha B = \bigcup \{A\colon A \cap B \neq \varnothing \text{ and } A \in \alpha\}.$$

There are two simplicial complexes associated with α. The *first* is the total complex, denoted X_α, whose simplexes are those finite subsets s of X such that for some $A \in \alpha$, s is a subset of A. The *second* is the simplicial complex called the nerve, $N(\alpha)$, of α. Its simplexes are the finite subsets s of α (or of the index set of α) such that the set $\bigcap\{A\colon A \in s\}$ is nonempty.

The *support* of a chain c in $C_*(X_\alpha)$ (coefficients in a field Q) is defined to be the smallest closed subset W of X such that c is a chain of $C_*(W_\alpha)$. If c is a chain in $C_*(N(\alpha))$, the *carrier* of c is the smallest subfamily β of α such that c is a chain of $C_*(N(\beta))$ (β need not be a covering of X for this to make sense). Let $\sup(c)$ be the union of those sets in the carrier of c.

According to Dowker [1] the two chain complexes $C_*(X_\alpha)$ and $C_*(N(\alpha))$ are chain homotopy equivalent; in fact his methods show that the following theorem is true.

THEOREM (DOWKER). *Let α be a family of subsets of X. Then there are chain maps*

$$k^\alpha\colon C_*(N(\alpha)) \to C_*(X_\alpha) \quad \text{and} \quad l^\alpha\colon C_*(X_\alpha) \to C_*(N(\alpha))$$

which are chain homotopy inverse to each other and which satisfy the two support conditions:

1. $\sup(k^\alpha(c)) \subset \sup(c)$, *for* $c \in C_*(N(\alpha))$,
2. $\sup(l^\alpha(c)) \subset \mathrm{St}_\alpha \sup(c)$, *for* $c \in C_*(X_\alpha)$.

DEFINITION 1. A compact space X is said to be "Q-simplicial" if for any open covering α of X there exists an $\alpha^\#$ which is a finite covering of X by open subsets such that $\alpha^\#$ refines α ($\alpha^\# > \alpha$) and such that for any open covering β of X there is a chain map ω: $C_*(X_{\alpha^\#}) \to C_*(X_\beta)$ such that $\sup(\omega(c)) \subset \mathrm{St}_\alpha(\sup(c))$, for $c \in C_*(X_{\alpha^\#})$.

This is the form of Q-simplicial which occurs in [2]. There is an equivalent form for nerves which we give below.

DEFINITION 2. A compact space X is said to be "Q-simplicial" if for any open covering α of X there exists an α^* which is a finite covering of X by open subsets such that $\alpha^* > \alpha$ and such that for any open covering β of X there is a chain map τ: $C_*(N(\alpha^*)) \to C_*(N(\beta))$ such that $\sup(\tau(c)) \subset \mathrm{St}_\beta(\mathrm{St}_\alpha(\sup(c)))$, for $c \in C_*(N(\alpha^*))$.

PROPOSITION. *Definitions* 1 *and* 2 *are equivalent.*

PROOF. Let X be a compact space. Let us prove that Definition 1 implies Definition 2. For an open covering α of X, let $\alpha^* = \alpha^\#$. For any open covering β of X, let ω be a chain map as in Definition 1. Then let τ:

$C_*(N(\alpha^*)) \to C_*(N(\beta))$ be the composition $\tau = l^{\beta}\omega k^{\alpha^*}$. One may compute straightforwardly that $\sup(\tau(c)) \subset \mathrm{St}_{\beta}(\mathrm{St}_{\alpha}(\sup(c)))$ for $c \in C_*(N(\alpha^*))$. Thus α^* satisfies the conditions of Definition 2, proving 1 implies 2.

To see that Definition 2 implies Definition 1, we again begin with an open covering α of X. By the compactness of X there is an open refinement γ of α such that for any subset B of X,

$$\mathrm{St}_{\gamma}(\mathrm{St}_{\gamma}(\mathrm{St}_{\gamma}(B))) \subset \mathrm{St}_{\alpha}(B). \tag{1}$$

Let $\alpha^{\#} = \gamma^*$, where γ^* is related to γ as α^* is to α in Definition 2. Let β be any open covering of X, and let β' be a common refinement of β and γ. Let τ: $C_*(N(\alpha^{\#})) \to C_*(N(\beta'))$ be a chain map such that $\sup(\tau(c)) \subset \mathrm{St}_{\beta'}(\mathrm{St}_{\gamma}(\sup(c)))$ for any chain $c \in C_*(N(\alpha^{\#})) = C_*(N(\gamma^*))$. Let π be the inclusion of $C_*(X_{\beta'})$ into $C_*(X_{\beta})$. Then define $\omega = \pi k^{\beta'}\tau l^{\alpha^{\#}}$. Since π and $k^{\beta'}$ reduce supports, then for a chain c in $C_*(X_{\alpha^{\#}}) = C_*(X_{\gamma^*})$,

$$\sup(\omega(c)) \subset \sup\left(\tau l^{\alpha^{\#}}(c)\right) \subset \mathrm{St}_{\beta'}\left(\mathrm{St}_{\gamma}\left(\mathrm{St}_{\gamma^*}(\sup(c))\right)\right). \tag{2}$$

Since β', γ and γ^* all refine γ, then by (1) and (2), $\sup\omega(c) \subset \mathrm{St}_{\alpha}(\sup(c))$. This proves that $\alpha^{\#}$ satisfies the conditions of Definition 1, completing the proof of the Proposition.

The following theorem is proven in [2] in a more general form including certain noncompact spaces.

THEOREM. *Suppose that X is a compact space which is Q-simplicial. Let f: $X \to X$ be a map such that the induced homology homomorphism*

$$f_*: H_*(X; Q) \to H_*(X; Q)$$

has an image which is finite dimensional as a vector space over Q. Then the Lefschetz number

$$\Lambda_Q(f) = \sum_n (-1)^n \mathrm{trace}(f_n)$$

is defined. If $\Lambda_Q(f) \neq 0$ then $f(x) = x$ for some x in X.

3. The main theorem.

THEOREM. *Suppose that G is a compact group and that f is a continuous function of G into itself. Let Q be any field of characteristic zero. If f_* has a finite dimensional image and $\Lambda_Q(f) \neq 0$, then there is a point x in G such that $f(x) = x$.*

PROOF. It suffices to show that G is Q-simplicial. We use Definition 2. From the first section G is a projective limit of compact Lie groups $G = \operatorname{proj}\lim G_{\mu}$. Any Lie group is triangulable so assume triangulations are given compatible with the homomorphisms of the inverse system. Let $\nu > \mu$ be indices of this system. Then there is a commutative diagram of onto simplicial maps

$$\begin{array}{ccc} \tilde{G}_\nu & \xrightarrow{\tilde{f}_{\mu\nu}} & \tilde{G}_\mu \\ p_\nu \downarrow & & \downarrow p_\mu \\ G_\nu & \xrightarrow{f_{\mu\nu}} & G_\mu \end{array}$$

such that $(\tilde{G}_\nu, p_\nu, G_\nu)$ and $(\tilde{G}_\mu, p_\mu, G_\mu)$ are finite sheeted covering spaces and such that $\tilde{f}_{\mu\nu}$ is a projection of a product space onto a factor space. Define a chain map $\tau_{\nu\mu}$: $C_*(G_\mu) \to C_*(G_\nu)$ as a composition of chain maps $\tau_{\nu\mu} = p_\nu \tau_1 \tau_2$ where τ_1: $C_*(\tilde{G}_\mu) \to C_*(\tilde{G}_\nu)$ is any simplicial approximation to a cross section $\tilde{G}_\mu \to \tilde{G}_\nu$, and τ_2: $C_*(G_\mu) \to C_*(\tilde{G}_\mu)$ is defined by $\tau_2(s) = n^{-1}\Sigma_{i=1}^n s_i$ where s is an oriented simplex of G_μ and where $s_1, \ldots, s_n$ are the oriented simplexes of $\tilde{G}_\mu$ which cover s. The number n is the same for every simplex, and it is clear that τ_2 is a chain map. Since $\tilde{f}_{\mu\nu}\tau_1$ = identity, and $p_\mu \tau_2$ = identity, then

(3) $$f_{\mu\nu}\tau_{\nu\mu} = \text{identity of } G_\mu.$$

It remains to show that G is Q-simplicial. Let $G_\mu^{(m)}$ be the mth barycentric derived complex of G_μ. If v is a vertex of $G_\mu^{(m)}$, then let St(v) be the star of v in $|G_\mu^{(m)}|$. Let f_μ: $G \to G_\mu = |G_\mu^{(m)}|$ be the limit homomorphism, and let

$$\alpha(\mu, m) = \left\{ f_\mu^{-1}(\mathrm{St}(v)) \colon v \text{ is a vertex of } G_\mu^{(m)} \right\}.$$

This is the inverse image under f_μ of the star covering of $|G_\mu^{(m)}|$ and as such is an open covering of G. The family of all such coverings (as μ and m range over their possible values) is cofinal in the directed set of all open coverings of G. Furthermore the nerve $N(\alpha(\mu, m))$ is naturally identified with $G_\mu^{(m)}$. Let c be a chain in $C_*(N(\alpha(\mu, m)))$, and write $v \in c$ if v is a vertex of a simplex s which occurs with nonzero coefficient. If we regard v as a vertex of $G_\mu^{(m)}$, then St(v) makes sense, and

(4) $$\sup(c) = f_\mu^{-1}\Big(\bigcup_{v \in c} \mathrm{St}(v) \Big).$$

It is clear that $\tau_{\nu\mu}$ induces a chain map $\tau_{\nu\mu}^{(m)}$: $C_*(G_\mu^{(m)}) \to C_*(G_\nu^{(m)})$; further

$$\sup\big(\tau_{\nu\mu}^{(m)}(c)\big) = \sup(c).$$

This is so, since by (3), $\sup(c) = \sup(f_{\mu\nu}\tau_{\nu\mu}^{(m)}(c))$ and $f_{\mu\nu}$ does not change supports (by (4) and the equality $f_\mu = f_{\mu\nu}f_\nu$).

For an open covering α of G, we must show how to choose α^* satisfying the conditions of Definition 2. Choose any μ and m such that $\alpha(\mu, m)$ refines α, and let $\alpha^* = \alpha(\mu, m)$. For any open covering β of G we must define a chain map τ: $C_*(N(\alpha^*)) \to C_*(N(\beta))$ such that $\sup(\tau(c)) \subset \mathrm{St}_\beta \mathrm{St}_\alpha(\sup(c))$. In fact we shall show how to define τ so that the stronger statement (5) holds.

(5) $$\sup(\tau(c)) \subset \mathrm{St}_\beta(\sup(c)).$$

Let ν and n be such that $n \geqslant m$, $\nu > \mu$, and $\alpha(\nu, n) > \beta$. Define τ as the composition

$$\tau = \pi \circ \mathrm{Sd} \circ \tau_{\nu\mu}^{(m)}$$

where π: $C_*(N(\alpha(\nu, n))) \to C_*(N(\beta))$ is induced by a projection of coverings, and Sd is the subdivision chain map

$$\begin{array}{ccc} \mathrm{Sd}: C_*(G_\nu^{(m)}) & \longrightarrow & C_*(G_\nu^{(n)}) \\ \| & & \| \\ C_*(N(\alpha(\nu, m))) & & C_*(N(\alpha(\nu, n))) \end{array}$$

Since $\tau_{\nu\mu}^{(m)}$ and Sd preserve supports and π satisfies $\sup(\pi(c)) \subset \mathrm{St}_\beta(\sup(c))$, then (4) holds. This completes the proof of the theorem.

CONCLUDING COMMENT. It would seem that any compact coset space G/M of a compact group G by a closed subgroup M would satisfy the Lefschetz fixed point theorem, but this question remains open.

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