A SUPERPOSITION THEOREM FOR BOUNDED CONTINUOUS FUNCTIONS

STEPHEN DEMKO

ABSTRACT. It is shown that there exist $2n + 1$ real valued, continuous functions $\phi_0, \ldots, \phi_{2n}$ defined on $\mathbb{R}^n$ such that every bounded real valued continuous function on $\mathbb{R}^n$ is expressible in the form $\sum_{i=0}^{2n} \circ \phi_i$ for some $g \in C(\mathbb{R})$. Extensions to some unbounded functions are also made.

The Kolmogorov superposition theorem states, in the form of Lorentz [4], that there exist $2n + 1$ real valued functions $\phi_1, \ldots, \phi_{2n+1}$ continuous on the closed unit cube, $I^n$, in $\mathbb{R}^n$ such that for any $f \in C(I^n)$ there is $g \in C(\mathbb{R})$ such that

\[
 f(x) = \sum_{i=1}^{2n+1} g(\phi_i(x)) \quad \text{for all } x \in I^n.
\]

In fact one may take $\phi_i$ to be of the form $\phi_i(x_1, \ldots, x_n) = \sum_{p=1}^{n} e^{\psi_i(x_p)}$ where $\psi_i \in C(I)$. The various proofs of this result, e.g., [2], [3], [4], all make use of the uniform continuity of $f$. Recently, Doss [1], has proven the following theorem.

THEOREM. There are functions $\psi_1, \ldots, \psi_{2n-1}, \phi_1, \ldots, \phi_{2n+1} \in C(\mathbb{R}^n)$ such that for any $f \in C(\mathbb{R}^n)$ there exist $g, h \in C(\mathbb{R})$ such that

\[
 f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{i=1}^{2n+1} g(\phi_i(x)) \quad \text{for all } x \in \mathbb{R}^n.
\]

Our main result is that there is a choice of $\phi_1, \ldots, \phi_{2n+1}$ such that for bounded functions (and some unbounded ones) we may take $h \equiv 0$. It has as an immediate corollary that every $f \in C(\mathbb{R}^n)$ can be represented in terms of $\phi_1, \ldots, \phi_{2n+1}$, the tangent function, and a function of a single variable. In the spirit of Hedberg [2], we obtain the existence of the $\phi_i$'s by means of a category argument. We then show that if $f$ has compact support, then the representing function $g$ must decay exponentially to zero away from the support of $f$. Paracompactness of $\mathbb{R}^n$ enables us to complete the proof. The norm used on functions is always the uniform norm on $\mathbb{R}^n$ unless stated otherwise.

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Throughout this paper \( n > 2 \) is a fixed but arbitrary integer and for \( m > 0 \) we define the following subsets of \( \mathbb{R}^n \): \( B_m = \{ x: \| x \| < m \} \), \( S_m = \{ x: \| x \| = m \} \), and \( R_m = \{ x: m - 1 < \| x \| < m \} \); here we use the \( l_\infty \) norm. We now recall the cubes of Kolmogorov, [3], [4].

**Lemma 1.** There exist closed, bounded cubes \( \{ c^q_k: 1 < q < 2n + 1, k > 1, i \in \mathbb{N} \} \) and strictly decreasing null sequences \( \{ a_k \}_{k > 1}, \{ b_k \}_{k > 1} \) such that

1. For fixed \( k \) each point of \( \mathbb{R}^n \) is contained in at least \( n + 1 \) of the cubes \( \{ c^q_k: 1 < q < 2n + 1, i \in \mathbb{N} \} \).
2. For fixed \( k \) and \( q \), the cubes \( \{ c^q_k: i \in \mathbb{N} \} \) are disjoint.
3. \( \text{diam}(c^q_k) < a_k \) for all \( i, q \).
4. \( \min_i H(c^q_k, c^{q'}_k) < b_k \) for all \( j, q \) where \( H(A, B) \) denotes the Hausdorff distance between \( A \) and \( B \).

The next result shows that the families of cubes can be separated by functions with prescribed ranges.

**Lemma 2.** There are \( 2n + 1 \) functions \( \phi_1, \ldots, \phi_{2n+1} \) in \( C(\mathbb{R}^n) \) such that

1. for \( 1 < q < 2n + 1 \) and \( m > 1 \), \( \phi_q(R_m) = [m, m + 2] \) and \( \phi_q(S_m) = [m + 1, m + 2] \).
2. for each \( N > 1 \), there is a \( k > N \) such that the sets \( \phi_q(c^q_k), 1 < q < 2n + 1, i \in \mathbb{N} \) are mutually disjoint.

**Proof.** The set \( M = \{ (\psi_1, \ldots, \psi_{2n+1}) \in [C(\mathbb{R}^n)]^{2n+1}: \) each \( \psi_q \) satisfies (a) \} is a nonempty complete metric space if it is given the product-compact-open topology. For \( k > 1 \) let \( \emptyset_k = \{ (\psi_q) \in M: \) the sets \( \psi_q(c^q_k), 1 < q < 2n + 1, i \in \mathbb{N} \) are mutually disjoint\}. It can be verified that for each \( N, \cup_{k > N} \emptyset_k \) is open and dense in \( M \). By the Baire category theorem \( \cap_{N > 1} \cup_{k > N} \emptyset_k \) is nonvoid; this is statement (b). Q.E.D.

**Lemma 3.** Let \( f \in C(\mathbb{R}^n) \) with \( \text{supp } f \subseteq \text{int} (\cup_{j=0}^j R_{m+j}) \) and let \( n(n + 1)^{-1} < \theta < 1 \). There exists \( g \in C(\mathbb{R}) \) such that

1. \( \| g \| < (n + 1)^{-1} \| f \| \),
2. \( |f(x) - \sum g(\phi_q(x))| < \theta \| f \| \), for all \( x \in \mathbb{R}^n \) and
3. \( \text{supp } g \subseteq [m - 1, m + l + 3] \).

**Proof.** Following Lorentz [4, p. 173], we let \( \epsilon > 0 \) be such that \( n(n + 1)^{-1} + \epsilon < \theta \). Let \( N \) be so large that if \( \| x - y \| < a_N \) (cf. Lemma 1), then \( |f(x) - f(y)| < \epsilon \| f \| \). Let \( k > N \) be such that the sets \( \phi_q(c^q_k), 1 < q < 2n + 1, i \in \mathbb{N} \), are mutually disjoint. We define \( g \) to be constant on each set \( \phi_q(c^q_k) \) the constant being the value of \( (n + 1)^{-1} f \) at the center of \( c^q_k \). Extending \( g \) linearly to all of \( \mathbb{R} \) we see that (a) is satisfied. The proof of (b) follows as in [4]. To prove (c) note that \( g(\phi_q(c^q_k)) \neq \{0\} \) only if \( c^q_k \cap (\cup_{j=0}^j R_{m+j}) = \emptyset \) and use part (a) of Lemma 2.

The next result shows that if \( f(x) \) has compact support, then it can be represented in the form \((*)\) where the function \( g(x) \) decays to zero exponentially as \( x \to \infty \).
Lemma 4. Let $f \in C(R^n)$ with $\text{supp} f \subseteq \text{int}(R_m \cup R_{m+1})$ and let $\theta$ be as in Lemma 3. There exists $g \in C(R)$ and a constant $C$ depending on only $\theta$ such that

(a) for all $x \in R^n, f(x) = \sum g(\phi_i(x))$ and 
(b) $\|g\|_{L^\infty[0:k+1]} < C\|\theta^{|m-k|}\|$ for all $k > 0$.

Proof. Let $g_1$ satisfy the conditions of Lemma 3 and define $f(x) = f(x) - \sum_i g_1(\phi_i(x))$. Note that $\text{supp} f_1 \subseteq \cup_{j=3}^{m+k} R_{m+j}$. In general, if $f_k$ is defined, we let $g_{k+1}$ be such that $\|g_{k+1}\| < (n+1)^{-1}\|f_k\|$ and $|f_k(x) - \sum_i g_{k+1}(\phi_i(x))| < \theta\|f_k\|$ and define $f_{k+1}(x) = f_k(x) - \sum_i g_{k+1}(\phi_i(x))$. As in [4], we have $\|f_k\| < \theta^k\|f\|$ and $\|g_{k+1}\| < \theta^k\|f\|$ which yield (a). By Lemma 3 the $g_k$'s can be chosen so that $\text{supp} f_k \subseteq \cup_{j=3}^{m+k} R_{m+j}$ and $\text{supp} g_{k+1} \subseteq [m - 3k - 1, m + 3k + 4] \cap [0, \infty)$. Consequently, on any interval of the form $[0, m - 3k - 1]$ or $[m + 3k + 4, \infty)$ the norm of $g$ is bounded by $\|f\|\Sigma_{m=k+1}^{\infty}\theta^r$. This implies (b). Q.E.D.

Theorem 1. Let $f \in C(R^n)$ be bounded. There exists $g \in C(R)$ such that for all $x \in R^n, f(x) = \sum_{m=0}^{\infty} g_m(\phi_i(x))$.

Proof. Since $R^n$ is paracompact, we may write $f(x) = \sum_{m=0}^{\infty} f_m(x)$ where $\text{supp} f_m \subseteq \text{int}(R_m \cup R_{m+1})$ and $\|f_m\| < \|f\|_{L^\infty(R_m \cup R_{m+1})}$. Let $g_m$ satisfy the conclusions of Lemma 4 for $f_m$. So,

$$f(x) = \sum_{m=0}^{\infty} \sum_{i=1}^{2^n+1} g_m(\phi_i(x)), \quad x \in R^n,$$

and

$$\|g_m\|_{L^\infty[0:k+1]} < C\|\theta^{|m-k|}\|f\|_{L^\infty(R_m \cup R_{m+1})}.$$

By the Weierstrass $M$-test, $\sum_{m=0}^{\infty} g_m \in C[R, k+2]$ for all $k > 0$. Therefore, $\sum_{m=0}^{\infty} g_m \in C(R)$ and $f(x) = \sum_{m=0}^{\infty} (\sum_{i=1}^{2^n+1} g_m(\phi_i(x)))$. Q.E.D.

Corollary 1. Let $f \in C(R^n)$ and assume that $\sum_{m=0}^{\infty} \theta^n \|f\|_{L^\infty(R_m)}$ converges for some $n(n+1)^{-1} < \theta < 1$, then there is $g \in C(R)$ such that

$$f = \sum_{i=1}^{2^n+1} g \circ \phi_i.$$

Proof. The assumptions insure that the series $\sum_{m=0}^{\infty} g_m$ converges uniformly on compact subsets of $R$.

We have not been able to extend Theorem 1 to all unbounded functions. However, it is possible to represent unbounded functions by nonlinear superpositions of the $\phi_i$'s.

Corollary 2. There is a constant $K > 0$ such that for $f \in C(R^n)$, there exists $g \in C(R)$ with $\|g\| < K$ and $f(x) = \tan(\sum_{i=1}^{2^n+1} g(\phi_i(x)))$ for all $x \in R^n$.

Proof. Since $f(x) = \tan(\text{Arc tan } f(x))$, apply Theorem 1 to $\text{Arc tan } f(x)$.

Remark. Theorem 1 can be extended to more general spaces than $R^n$. For example, let $\Omega$ be an open connected subset of $R^n$. In the arguments used
above, replace $R_m$ by $\{ x \in \Omega : 1/m < \text{dist}(x, \partial \Omega) < 1/(m-1) \}$, $S_m$ by $\{ x \in \Omega : \text{dist}(x, \partial \Omega) = 1/m \}$ and $B_m$ by $\{ x \in \Omega : 1/m < \text{dist}(x, \partial \Omega) \}$.

References


School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332