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THE NUMERICAL RANGE AND THE ESSENTIAL NUMERICAL RANGE

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ABSTRACT. A simple proof is given of Lancaster's theorem that the convex hull of the numerical and essential numerical ranges of a Hilbert space operator is the closure of the numerical range.

For an operator A on a complex separable Hilbert space $\mathcal{H}$ the relation between the numerical range $W(A)$ of A, its closure, and the essential numerical range $W_e(A)$ was found by John Lancaster [3] who proved

THEOREM. $W(A)^- = \operatorname{conv}(W(A) \cup W_e(A))$.

Here "conv" denotes "convex hull", not "closed convex hull", so that one consequence of the theorem is that the numerical range is closed if and only if it contains the essential numerical range.

The purpose of this note is to present a proof of Lancaster's theorem that seems conceptually simpler than the original.

Recall that an element A of a unital complex Banach algebra $\mathcal{A}$ has a numerical range $V(A)$ which by definition consists of the complex numbers $f(A)$ with f a state on $\mathcal{A}$ (that is, a linear functional with $f(1) = 1 = \|f\|$) [1], [4]. When $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$, the set $V(A)$ is the closure of the familiar numerical range $W(A) = \{(A\xi,\xi): \xi \in \mathcal{H}, \|\xi\| = 1\}$. And if π denotes the quotient map from $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$, then $V(\pi(A))$ is the *essential numerical range* $W_e(A) = \bigcap W(A + K)^-$, the intersection being taken over all K in the ideal $\mathcal{K}$ of compact operators on $\mathcal{H}$ [4].

Thus $W(A)^-$ (resp. $W_e(A)$) is obtained by evaluation of all the states of $\mathcal{B}(\mathcal{H})$ (that vanish on $\mathcal{K}$) at the operator A. Now it is a fact due to Dixmier [2] that every state f on $\mathcal{B}(\mathcal{H})$ has the form $f = \alpha f_0 + (1 - \alpha) f_T$ where $0 \leqslant \alpha \leqslant 1$, f_0 is a state that annihilates $\mathcal{K}$, and f_T is the state induced by a nonnegative trace class operator T: $f_T(X) = \operatorname{trace}(XT)$ for $X \in \mathcal{B}(\mathcal{H})$. It follows at once that $W(A)^-$ is the convex hull of the essential numerical range and the "trace class numerical range" consisting of the numbers $f_T(A)$. To complete the proof of the theorem it suffices therefore to show that this new set is just $W(A)$.

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Any such T has spectral decomposition $T = \Sigma\, \lambda_n(\cdot, e_n)e_n$ where the e_n are an orthonormal set, $\lambda_n \geqslant 0$, and $\Sigma\, \lambda_n = 1$. Consequently

$$f_T(A) = \Sigma\, \lambda_N(Ae_n, e_n)$$

belongs to $W(A)$ because any convex subset of the plane (or $\mathbf{R}^n$) contains convex combinations of its countable subsets. (This latter fact is apparently well known to specialists. Its proof is a routine convexity argument.)

NOTE ADDED (December 1976). Lancaster's theorem has been generalized from numerical ranges to matrix ranges by J. Bunce and N. Salinas in Theorem 3.7 of their preprint *Completely positive maps on C^*-algebras and the left matricial spectra of an operator.*

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