SCHUR MULTIPLIERS OF SOME
FINITE NILPOTENT GROUPS

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Abstract. Let $B$ denote the Burnside group, $B(p^n, d)$ and let $G = B/B_k$ where $p$ is a prime and $1 < k < p$. We show that the Schur multiplier, $M(G)$, is a direct power of $\Psi(k, d)$ cyclic groups, each having order $p^n$, where $\Psi(k, d) = k^{-1} \sum_{\mu(k|n)} d^\mu$. (This is Witt's formula for the rank of $F_k/F_{k+1}$ where $F$ is free on $d$ generators.) In addition we can show that $M(B(3, d))$ is elementary abelian of exponent 3 and rank $2(f) + 4(f) + 3(f)$.

1. Notation and a preliminary result. For any group $G$, $M(G)$ will be the Schur multiplier of $G$. The minimal number of generators for $G$ will be denoted $d(G)$. When $G$ is abelian we will sometimes refer to $d(G)$ as the rank of $G$ and write $rk(G)$. $Z(G)$ will be the center of $G$ and $G'$ will be the commutator subgroup of $G$. $G_k$ will be the $k$th term of the lower central series $G_1 = G$, $G_2 = G' = (G, G)$, $G_3 = (G_2, G)$, . . . .

Throughout this paper $d$ will be a fixed positive integer, $p$ will be a fixed prime and $q$ will be $p^a$ for some fixed positive integer $a$. $F$ will be a free group on a set $\{x_1, x_2, \ldots, x_d\}$ of $d$ generators, and $\Psi(k, d)$, the rank of $F_k/F_{k+1}$, will be abbreviated $\Psi(k)$. $B(q, d)$ will be the Burnside group with exponent $q$ on $d$ generators.

I thank N. Blackburn for showing me a proof of the following generalization of a theorem of R. C. Lyndon [7].

Theorem 1.1. Let $B = B(q, d)$. Then $B_k/B_{k+1}$ is a direct sum of $\Psi(k)$ cyclic groups of order $q$, for $k < p$.

Proof. We claim that $F_k \cap F^q F_{k+1} = F^q F_{k+1}$ for $k < p$, hence $B_k/B_{k+1} \cong F^q F_{k+1}/F^q F_k \cong F^q F_{k+1}/F_{k+1}$.

To prove the claim we show by induction on $j$, for $1 < j < k$, that $F_j \cap F^q F_{k+1} \subseteq F^q F_{k+1}$. Given $w \in F_j \cap F^q F_{k+1}$, then $w \in F_{j-1} \cap F^q F_{k+1}$, so by induction $w \in F^q_{j-1} F_{k+1}$. After moving the factors from $F_{k+1}$ to the right, we can write $w = a_1^{q_j} \cdots a_r^{q_j} x$ where $a_i \in F_{j-1}$ and $x \in F_{k+1}$. Now consider $(a_1 a_2 \cdots a_r)^q$ and apply Theorem 12.3.1 of [2] to get

$$(a_1 a_2 \cdots a_r)^q = a_1^{q_j} a_2^{q_j} \cdots a_r^{q_j} c_1^{q_j} \cdots c_r^{q_j}$$

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where \( y \in F_{k+1} \), each \( c_i \in F_j \) and \( q \) divides each \( e_i \) since \( k < p \). Observe that \( wx^{-1} = a_1^q \cdots a_r^q \in F_j \), so \( (a_1a_2 \cdots a_r)^q \in F_j \). Since \( F/F_j \) is torsion free, \( a_1a_2 \cdots a_r \in F_j \). But then since \( q \) divides each \( l_i \), we get \( a_1^q \cdots a_r^q \in F_j F_{k+1} \).

2. The exponent of \( M(G) \).

**Theorem 2.1.** Let \( B = B(q, d) \), \( G = B/B_k \) where \( k < 2p \) and \( Q = G/N \) where \( N \subseteq G_l \) and \( l > k - p \). Then \( M(Q) \) has exponent dividing \( q \).

**Proof.** We may assume that \( d \) is chosen so that \( d(Q) = d(G) = d \) and that \( k \) is chosen so that class \( Q = \text{class } G = k - 1 \).

Consider an exact sequence \( M \rightarrow H \rightarrow Q \) where \( M \subseteq H \cap Z(H) \). Then \( H \) is a finite \( p \)-group with \( d(H) = d \) and class \( H < k \). Let \( \{z_1, z_2, \ldots, z_d\} \) be a set of generators for \( Q \) and let \( \{h_1, h_2, \ldots, h_d\} \) be a set of generators for \( H \) with \( \pi h_i = z_i \). For any commutator \( \tilde{c} \) in \( Q \) with entries from \( \{z_i\} \) we will let \( c \) denote the corresponding commutator in \( H \) with each \( z_i \) replaced by \( h_i \).

Observe that for any \( h_i, h_i^q \in M \), and thus is central. Hence given \( c = (h_1, \ldots, h_d) \), we have \( (cH_{j+1})^q = (h_1^q, \ldots, h_j^q)H_{j+1} = H_{j+1} \). It follows that \( H_j/H_{j+1} \) has exponent dividing \( q \). We wish to show that \( H_{k-p+1} \) has exponent dividing \( q \). We show by induction on \( \lambda \), for \( 0 < \lambda < p \), that \( H_{k-\lambda} \) has exponent dividing \( q \). Given \( \lambda < p \), write \( r = k - \lambda \), so that \( r + p > k \). Then for any commutator \( c = (h_1, \ldots, h_d) \in H \) we use, e.g., Lemma H2 of [9] to write

\[
1 = (h_1^q, \ldots, h_d^q) = c^qc_1^q \cdots c_d^q
\]

where the \( c_i \) are all commutators with \( r < \text{wtc}_i < k \). Since \( p + r > k \), \( q \) divides each \( e_i \). By induction the \( c_i \) have exponent dividing \( q \), so \( c \) has exponent dividing \( q \) as well. For an arbitrary element \( x \) of \( H_{k-\lambda} \), we may write \( x \) as a product of commutators of weight at least \( k - \lambda \) and apply Theorem 12.3.1 of [2] together with the above result for commutators to show that \( x \) has exponent dividing \( q \).

Thus to prove the theorem it suffices to show that \( M \subseteq H_{k-p+1} \). Since \( l > k - p + 1 \) and \( p > k - p + 1 \) we are done if we show either that \( M \subseteq H_l \) or that \( M \subseteq H_p \). We find the remainder of the argument easier to follow if we separate the cases \( l < p \) and \( l > p \).

**Case (i) \( l < p \).** We show in this case that \( M \subseteq H_l \). For each \( j \), choose a basis \( \{\tilde{c}_1Q_{j+1}, \ldots, \tilde{c}_rQ_{j+1}\} \) for \( Q_j/Q_{j+1} \), taking \( \{z_1Q_2, z_2Q_2, \ldots, z_dQ_2\} \) as the choice for \( Q/Q_2 \). Fix a weight preserving order of all of the \( \tilde{c} \) and relabel according to this order, so that (up to the choice of bases and the choice of the order) each element \( w \) of \( Q \) has a unique expression \( w = \prod \tilde{c}_i \) where \( \tilde{c}_i \in Q_j, \tilde{c}_iQ_{j+1} \) has order \( \beta_i \) in \( Q_j/Q_{j+1} \) and \( 0 < \alpha_i < \beta_i \).

For \( j < l \), \( Q_j/Q_{j+1} \cong B_j/B_{j+1} \), hence \( r(j) = \Psi(j) \) and \( Q_j/Q_{j+1} \) is homocyclic of exponent \( q \). We have seen above that \( H_j/H_{j+1} \) has exponent dividing \( q \). Since \( Q_j/Q_{j+1} \) is an image of \( H_j/H_{j+1} \), the latter must have rank \( \Psi(j) \) and exponent \( q \) for \( j < l \). The corresponding sets
{c_1H_{j+1}, \ldots, c_{\Psi(l)}H_{j+1}} form bases for H_j/H_{j+1} when j < l. Choose, as well, bases for H_j/H_{j+1} when j > l. Order those c_i with weight less than l in the same way as the \tilde{c}_i were ordered, and extend this order to a weight preserving order of all of the c_i's. Then each element h of H has a unique expression $h = \prod c_i^{\alpha_i}$ where $c_i \in H_j$, $c_iH_{j+1}$ has order $\beta_i$ in $H_j/H_{j+1}$ and $0 < \alpha_i < \beta_i$. If we apply the quotient map $\pi$ to h, we get

$$\pi h = \prod_{w t c_i < l} c_i^{\alpha_i} = \prod_{w t c_i > l} \tilde{c}_i^{\alpha_i} \cdot \prod_{w t c_i > l} c_i^{\alpha_i}$$

so that the unique expression for $\pi h$ has $\tilde{\alpha}_i = \alpha_i$ for $w t c_i < l$ and $\tilde{\alpha}_i$ equal to some $\alpha_i'$ when $w t c_i > l$. If $h \in M$, then $\pi h = 1$, so $\tilde{\alpha}_i = 0$ for all $i$. Hence $\alpha_i = 0$ for $w t c_i < l$ and $h \in H_j$.

Case (ii) $l > p$. The argument here is the same as in Case (i) except we observe that $Q_j/Q_{j+1} = B_j/B_{j+1} \cong H_j/H_{j+1}$ for $j < p$ and show that $M \subseteq H_p$.

Corollary 2.2. Let $B = B(q, d)$, $G = B/B_k$ where $p < k < 2p$ and let $M \twoheadrightarrow H \twoheadrightarrow G$ be exact with $M \subseteq H' \cap Z(H)$. Then $M \subseteq H_p$.

Corollary 2.3. Let $B = B(q, d)$ and $G = B/B_k$ where $k < 2p$. Then $M(G)$ has exponent dividing $q$.

Corollary 2.4. If $Q$ is any group of exponent 3, then $M(Q)$ has exponent dividing 3.

Proof. Let $d = d(Q)$ and $B = B(3, d)$. Then $B = B/B_4$ (see [2]) and $Q = B/N$ for some $N \subseteq B_2$.

Corollary 2.4 is implicit in Remark 2.8 of Jones [5] since any group of exponent 3 satisfies the 2nd Engel condition.

We do not have any example to show that the condition on $k$ in Theorem 2.1 is necessary. The example of Bayes, Kautsky, and Wamsley [1] of a group $G$ with exponent 4 with $M(G)$ having exponent 8 shows that the hypothesis on $l$ cannot be dropped.

3. The multiplier of groups having exponent $q$.

Theorem 3.1. If $B = B(q, d)$ and $G = B/B_k$ where $1 < k < p$, then $M(G)$ is homocyclic with exponent $q$ and rank $\Psi(k)$.

Proof. Observe first that $B_k/B_{k+1} \twoheadrightarrow B/B_{k+1} \twoheadrightarrow G$ is exact with $B_k/B_{k+1}$ in both the commutator and central subgroups of $B/B_{k+1}$. It follows from a theorem of Schur, see [3, V, Satz 23.5], that $B_k/B_{k+1}$ is an epimorphic image of $M(G)$. Then by Theorem 1.1 we see that $\text{rk } M(G) > \Psi(k)$ and $|M(G)| > q^{\Psi(k)}$.

By Corollary 2.3, $M(G)$ has exponent dividing $q$, so it will suffice to show that $\text{rk } M(G) < \Psi(k)$. To this end we exhibit a presentation for $G$.

Let $\{c_iF_{k+1}, \ldots, c_{\Psi(k)}F_{k+1}\}$ be a basis for $F_k/F_{k+1}$. Then we claim that $G$ has a presentation

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Let $\widetilde{G}$ be the group having this presentation. The obvious function $\widetilde{G} \to G$ preserves relations, and hence is a group epimorphism. $\widetilde{G}$ is nilpotent of class $k - 1$ and $\text{rk}\, \widetilde{G}_j/\widetilde{G}_{j+1} \leq \Psi(j) = \text{rk}\, G_j/G_{j+1}$. We argue as in Theorem 2.1 that each $\widetilde{G}_j/\widetilde{G}_{j+1}$ has exponent dividing $q$. It follows that $|\widetilde{G}_j/\widetilde{G}_{j+1}| < |G_j/G_{j+1}|$, so that $|\widetilde{G}| < |G|$.

Having this presentation for $G$, it follows, as in [3, V, Satz 25.2], that $\text{rk}\, M(G) \leq \Psi(k)$. Since $G$ is nilpotent of class $k - 1$, the relators in $F_{k+1}$ contribute nothing to the number of generators of $M(G)$.

It would be of interest to know if $F_{k+1}$ can be omitted from the set of relators in the above presentation. If so, the resulting presentation has deficiency $-\text{rk}\, M(G)$.

**Proposition 3.2.** If $B = B(q, d)$ and $G = B/B_p$ then $M(G)$ has exponent dividing $q$ and $\text{rk}\, M(G) \leq \Psi(p)$.

**Proof.** Let $M \to H \to G$ be exact with $M \subseteq H' \cap Z(H)$. By Corollary 2.2, $M \subset H_p$. But $H_{p+1} = 1$, so $\text{rk}\, H_p \leq \Psi(p)$.

Using results of Lyndon [7], one can easily show that $$\text{rk}\, M(G) > \Psi(p) + d - \left(\frac{p + d - 1}{p}\right)$$ for $G = B/B_p$.

The difficulty in improving the result here is that we do not have a convenient representation group to exhibit.

4. The multipliers of groups of exponent 3. Throughout this section we let $B = B(3, d)$. $B$ has been described by Levi and van der Waerden [6]. For details see, for example, [2]. Our present proofs of results in this section are tedious, but straightforward, and we condense or omit them.

**Lemma 4.1.** $B$ has presentations with $d$ generators and $r = d + 2(d) + 4(d) + 3(d)$ relators.

**Proof.** One such presentation is

\begin{align*}
\langle x_1, x_2, \ldots, x_d | &x_i^3 = 1, 1 \leq i \leq d, \\
(x_j, &x_i, x_i) = (x_j, x_i, x_j) = 1, 1 \leq i < j \leq d, \\
(x_k, &x_j, x_i, x_i) = (x_k, x_j, x_i, x_j) = (x_k, x_j, x_i, x_k) \\
&= (x_k, x_j, x_i)(x_k, x_i, x_j) = 1, 1 \leq i < j < k \leq d, \\
(x_l, &x_k, x_j, x_i) = (x_l, x_k, x_j, x_i) = (x_l, x_j, x_i, x_k) = 1, \\
&1 \leq i < j < k < l \leq d \rangle.
\end{align*}

**Lemma 4.2.** There is a group $H$ and a short exact sequence $M \to H \to B$ where $M \subseteq Z(H) \cap H'$ and $M$ is elementary abelian of exponent 3 and rank $r - d = 2(d) + 4(d) + 3(d)$.

The presentation in Lemma 4.1 can be regarded as a blueprint for
constructing such a representation group $H$. The details are tedious to verify.

**Theorem 4.3.** $M(B)$ is elementary abelian of exponent 3 and rank $r - d = 2(q^2) + 4(q^2) + 3(q^2)$.

**Proof.** This follows easily from Lemmas 4.1 and 4.2.

We remark that MacDonald [8] has previously computed the rank of $M(B(3,3))$ as an application of a computer algorithm.

For the case $q = 3$, we can improve on Proposition 3.2.

**Proposition 4.2.** Let $G = B/B_3$. Then $M(G)$ has exponent 3 and rank $\Psi(3)$.

Representation groups for $G = B/B_3$ are much easier to construct than those for $B$. We note, however, that, with a proper choice $H$ of a representation group for $B$, $H/H_4$ is a representation group for $G$.

As a final remark, we note that essentially all of the multipliers that we are obtaining are nontrivial and that this is not surprising in view of D. L. Johnson’s recent paper [4].

**References**


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