UNITARY PARTS OF CONTRACTIVE HANKEL MATRICES

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ABSTRACT. For a Hankel matrix \( H = (c_{j+k}) \) which is a contraction, necessary and sufficient conditions are obtained for the existence of a nontrivial unitary part, and an explicit description of this unitary part is given.

1. Introduction. In what follows, \( l^2 \) will denote the usual class of square-summable complex sequences, \( L^2 \) will denote the class of Lebesgue measurable functions on the unit circle in the complex plane, and \( H^2 \) will denote the Hardy (closed) subspace of \( L^2 \) consisting of those functions in \( L^2 \) whose Fourier coefficients vanish on the negative integers. Functions which differ only on zero sets will be considered equal.

With a sequence \((c_0, c_1, \ldots)\) in \( l^2 \) we associate the Hankel matrix \( H = (c_{j+k}) \), where \( j, k = 0, 1, 2, \ldots \), acting on \( l^2 \). If we define \( \phi(e^{i\theta}) = c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \ldots \), then such a matrix may be realized as an operator on \( H^2 \), defined by

\[
Hx(e^{i\theta}) = P_+ \phi(e^{i\theta})x(e^{-i\theta}),
\]

where \( P_+: L^2 \to H^2 \) is the orthogonal projection.

A result of Nehari [4] states that a Hankel matrix \( H \) is bounded if and only if there exists a function \( f \in L^\infty \) such that

\[
(1-1) \ c_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta})e^{-in\theta} d\theta,
\]

and, in this case, \( f \) can be chosen so that \( \|f\|_\infty = \|H\| \). Such a function \( f \) is called a minifunction of \( H \).

We shall consider Hankel matrices which are contractions (\( \|Hx\| \leq \|x\| \)) on the Hardy space \( H^2 \). If a Hankel matrix is a contraction, then, by the theorem of Nehari, we can find a minifunction \( f \in L^\infty \) such that \( \|f\|_\infty = \|H\| \leq 1 \) and \( Hx(e^{i\theta}) = P_+ f(e^{i\theta})x(e^{-i\theta}) \) for \( x \in H^2 \).

2. The unitary part of a Hankel contraction. Following Sz.-Nagy and Foiaș [5], we say that a contraction \( T \) on a Hilbert space \( K \) is completely nonunitary if \( T \) has no nontrivial reducing subspace \( N \) such that the restriction \( T|N \) of \( T \) to \( N \) is unitary. It is known [5, Theorem 1.3.2] that for any contraction \( T \) on \( K \) we can find a unique orthogonal decomposition \( K = M \oplus M_1 \) such that \( T|M \) is unitary and \( T|M_1 \) is completely nonunitary. It is not excluded that \( M \) or

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$M_1$ is possibly the subspace $\{0\}$. Furthermore, $M$ is given by $M = \{x \in K: \|T^n x\| = \|x\| = \|T^m x\|, n = 1, 2, \ldots \}$ and is called the unitary subspace of $T$. $T|M$ is called the unitary part of $T$.

If $\psi(e^{i\theta}) \in L^\infty$, then the corresponding bounded Toeplitz operator $T_\psi: H^2 \to H^2$ is defined by $T_\psi: x(e^{i\theta}) \mapsto P_+ \psi(e^{i\theta}) x(e^{i\theta})$ for $x \in H^2$. In [1] Goor proved that if $T_\psi$ is a Toeplitz contraction ($\|T_\psi\| < 1$, i.e., $|\psi(e^{i\theta})| < 1$ a.e.), then $T_\psi$ is completely nonunitary unless $\psi$ is a constant. This result may be used to obtain necessary and sufficient conditions for the existence of a nontrivial unitary part of a Hankel contraction. These conditions, together with a characterization of the unitary part, when it exists, are obtained as a corollary to the following.

**Theorem 2.1.** Let $H = (c_{j+k})$ be a Hankel contraction. Then $H$ will have a nontrivial unitary subspace $M$ only when there exists a minifunction $f(e^{i\theta})$ for $H$ such that

(2.1) $|f(e^{i\theta})| = 1$ a.e. (so that $\|H\| = 1$), and

(2.2) $f(e^{i\theta}) f(e^{-i\theta}) = k^2$ a.e. for some constant $k^2$, $|k| = 1$.

In such a case, the minifunction $f(e^{i\theta})$ is unique, and $M$ is given by the three equivalent expressions:

$$M = \left[ e^{i\theta} f(e^{i\theta}) H^2 \right]^\perp \cap H^2,$$

where the orthogonal complement is in $L^2$,

(2.4) $M = \{x \in H^2: H^* H x = x\}$,

(2.5) $M = \{x \in H^2: \overline{k} H x = x\} \oplus \{x \in H^2: \overline{k} H x = -x\}$.

**Proof.** Suppose that a Hankel contraction $H$ has a nontrivial unitary subspace. Then there exists some $x \neq 0$ in $H^2$ such that for $H$ and its adjoint $H^* = \overline{H}$ we have $\|H^* x\| = \|x\| = \|\overline{H}^* x\|$ for $n = 1, 2, \ldots$. Taking $n = 1$ gives that

$$\|x\| = \|H x\| = \|P_+ f(e^{i\theta}) x(e^{-i\theta})\| < \|f(e^{i\theta}) x(e^{-i\theta})\| < \|x(e^{-i\theta})\|.$$ 

This implies that $P_+ f(e^{i\theta}) x(e^{-i\theta}) = f(e^{i\theta}) x(e^{-i\theta})$, so that $H x = f(e^{i\theta}) x(e^{-i\theta}) \in H^2$. It also immediately follows that the minifunction $f(e^{i\theta})$ must therefore be unique, cf. [2, p. 863]. Furthermore, we may apply a well-known corollary of the F. and M. Riesz theorem [3, p. 52] to the equality

$$\|f(e^{i\theta}) x(e^{-i\theta})\| = \|x(e^{-i\theta})\|$$

to conclude that $|f(e^{i\theta})| = 1$ almost everywhere on the unit circle. This establishes (2.1).

Taking now $n = 2$, we get

$$\|x\| = \|H^2 x\| = \|H f(e^{i\theta}) x(e^{-i\theta})\| = \|P_+ f(e^{i\theta}) f(e^{-i\theta}) x(e^{i\theta})\| < \|f(e^{i\theta}) f(e^{-i\theta}) x(e^{i\theta})\| < \|x(e^{i\theta})\|,$$

so that

$$H^2 x = f(e^{i\theta}) f(e^{-i\theta}) x(e^{i\theta}) = T_\psi x,$$

where $\psi(e^{i\theta}) = f(e^{i\theta}) f(e^{-i\theta})$. Continuing, we obtain
But (2.7) together with its analogue for $H^{2n}$ (obtained by replacing $f(e^{i\theta})$ by $\tilde{f}(e^{i\theta})$) implies that the Toeplitz operator $T_\phi$ has a nontrivial unitary part. Therefore, by Goor's result, $f(e^{i\theta})f(e^{-i\theta}) = k^2$ for some constant $k$, $|k| = 1$. Hence, $k\tilde{f}(e^{i\theta}) = k\tilde{f}(e^{-i\theta})$, so that, by (1.1), $k_n x$ is real for all $n$.

Formulas (2.6) and (2.7) now reduce to

$$H^{2n+1}x(e^{i\theta}) = k^{2n}Hx(e^{i\theta}), \quad n = 0, 1, 2, \ldots,$$

$$H^{2n}x(e^{i\theta}) = k^{2n}x(e^{i\theta}), \quad n = 1, 2, \ldots,$$

valid for all $x$ in the unitary subspace of $H$. Similar expressions are easily obtained for $H^{2n+1}x(e^{i\theta})$ and $H^{2n}x(e^{i\theta})$.

The maximal subspace on which $H$ is unitary now becomes $M = \{x \in H^2: \|Hx\| = \|x\|\} = \{x \in H^2: H^*Hx = x\}$, giving (2.4). Furthermore, we then have $x \in M \iff f(e^{i\theta})x(e^{-i\theta}) \in H^2 \iff f(e^{-i\theta})x(e^{i\theta}) \perp e^{i\theta}H^2 \iff x(e^{i\theta}) \perp e^{i\theta}\tilde{f}(e^{-i\theta})\tilde{H}^2 \iff x \in \{e^{i\theta}f(e^{i\theta})H^2\} \cap H^2$, since $f(e^{-i\theta}) = k^2\tilde{f}(e^{i\theta})$. This establishes (2.3). Finally, since the matrix $kH$ is real, hence selfadjoint, with $\|kH\| = 1$, we get (2.5). This completes the proof.

As a result of the above theorem, we then obtain a characterization of those Hankel matrices having nontrivial unitary subspaces.

**Corollary 2.1.** Let $H$ be a Hankel contraction. Then a necessary and sufficient condition that $H$ have a nontrivial unitary subspace is that there exist a constant $k$, $|k| = 1$, such that $kH$ is real (hence selfadjoint) and that $M$ in (2.5) satisfy $M \neq \{0\}$.

Using a result of Sz.-Nagy and Foiaş for completely nonunitary contractions, we can define a functional calculus for functions $u \in \mathcal{H}^\infty$. In particular, if $B(H^2)$ is the space of bounded operators on $H^2$, we then have the following.

**Corollary 2.2.** Let $H$ be a Hankel contraction. If either $\|H\| < 1$ or $kH$ is not real for all $k \in \mathbb{C}$, then the map $u \mapsto u(H)$ from the Hardy space $\mathcal{H}^\infty$ into $B(H^2)$ defined by

$$u(H) = \text{strong lim}_{r \to 1^-} \sum_{k=0}^{\infty} a_k r^k H^k,$$

where $u(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$, is a contractive homomorphism of the algebra $\mathcal{H}^\infty$ into $B(H^2)$.

This follows from the above theorem and Theorem III.2.1 of [5].

Theorem 2.1 can also be used to establish a property of the point spectrum of a bounded Hankel matrix.

**Corollary 2.3.** Let $H$ be a bounded Hankel matrix. If $\lambda = \|H\|$ is an eigenvalue of $H$, then $H$ is real. Hence, if $aH$ is not real for any nonzero
(complex constant) $\alpha$, then $|\lambda| < \|H\|$ holds for all eigenvalues $\lambda$ of $H$ (if any).

**Proof.** Without loss of generality, we can suppose that $\|H\| = 1$. (In such a case, the associated eigenspace $M = \{y \in H^2: Hy = y\}$ is well-known to be a reducing subspace for the operator $H$ [5, Proposition 1.3.1].) If $y \neq 0$ satisfies $Hy = y$, then, by the proof of Theorem 2.1, $\overline{H} = \overline{\alpha}H$ and $H^2 y = \alpha y$ for some $|\alpha| = 1$. But since $H^2 y = y$, we get $\alpha = 1$, and hence $\overline{H} = \overline{\alpha}H = H$, which shows that $H$ must be real.

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**References**


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