A HELLY TYPE THEOREM ON THE SPHERE

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Abstract. This paper establishes a Helly type theorem for convex sets in $S_n$, the $n$-dimensional unit sphere.

1. Introduction. Let $S_n$ denote the $n$-dimensional unit sphere in Euclidean $(n + 1)$-dimensional space, $R^{n+1}$.

A subset of $S_n$ will be called convex if it is the intersection of $S_n$ with a convex cone with apex 0 in $R^{n+1}$.

A subset of $S_n$ will be called strongly convex if it is convex and does not contain antipodal points.

Thus a set consisting of two antipodal points is convex but not strongly convex.

For other definitions of convex sets in $S_n$ and for a survey of Helly type theorems see [2]. For standard notation and terminology see [3].

M. J. C. Baker has proven in [1] the following

THEOREM. For all positive integers $n$ and $t$, if the intersection of each $n + 1$ members of a family of at least $n + 1 + 2t$ strongly convex sets in $S_n$ is nonempty then the intersection of some $n + 1 + t$ members of the family is nonempty.

The purpose of this paper is to generalize Baker's result by proving

THEOREM A. Let $n$ and $t$ be positive integers and let $\mathcal{A}$ be a family of $n + 1 + t$ convex sets in $S_n$.

If every $n + 1$ members of $\mathcal{A}$ have nonempty intersection then the intersection of some $n + 1 + [t/2]$ members of $\mathcal{A}$ is nonempty.

Moreover, $n + 1 + [t/2]$ cannot be replaced by $n + [t/2] + 2$.

It is possible to improve Theorem A for strongly convex sets and $t = 2$ by establishing

THEOREM B. Let $n$ be an integer and let $\mathcal{A}$ be a family of $n + 3$ strongly convex sets in $S_n$. Suppose that every $n + 1$ members of $\mathcal{A}$ have nonempty intersection.

If $k$ is the number of different $n + 2$ membered subfamilies $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcap \mathcal{B} = \emptyset$ then $k \leq 2$.

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The proof of Theorem A is based on
(1) Properties of certain families of convex cones called nondegenerate families (N.D.F.'s), discussed in [4] and [5].
(2) The existence, or the nonexistence of certain $k$-neighborly polytopes in $R^n$; see [3].

All the necessary definitions and properties of N.D.F.'s and of neighborly polytopes are stated in §2.

A proof of Theorem A is given in §3.

The proof of Theorem B is similar to the proof of Theorem A and is therefore omitted.

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2. Nondegenerate families and neighborly polytopes. Let $\mathcal{G}$ be a nonempty finite family of convex cones with apex 0 in $R^n$.

The family $\mathcal{G}$ will be called a nondegenerate family (a N.D.F.) if each member of the family is of dimension $n$, the intersection of any two members of the family is at least of dimension $(n-1), \ldots$, the intersection of any $n$-members is at least of dimension 1 and the intersection of all members is $\{0\}$.

A subset $\mathcal{B}$ of $\mathcal{G}$ will be called a face of $\mathcal{G}$ if $\cap (\mathcal{G} \setminus \mathcal{B})$ is a subspace.

A subset $\mathcal{B}$ of $\mathcal{G}$ is a $k$-face of $\mathcal{G}$ if it is a face of $\mathcal{G}$ and if $\dim \cap (\mathcal{G} \setminus \mathcal{B}) = |\mathcal{B}| - k$. ($|\mathcal{B}|$ means the cardinality of $\mathcal{B}$.)

A relationship between N.D.F.'s and polytopes is given by

**Theorem C.** (i) If $\mathcal{G}$ is a N.D.F. in $R^n$ and $|\mathcal{G}| = n + t$ then there exists a $(t-1)$-polytope $P$ in $R^{t-1}$ such that

1. The lattice of faces of $P$ and the lattice of faces of $\mathcal{G}$, both ordered by the inclusion relation, are isomorphic. The isomorphism carries $k$-faces of $P$ to $(k+1)$-faces of $\mathcal{G}$.

(ii) For any $(t-1)$-polytope $P$ with $n + t$ vertices there exists a N.D.F. $\mathcal{G}$ in $R^n$ with $|\mathcal{G}| = n + t$ such that (1) is satisfied.

A proof of Theorem C is given in [5].

**Neighborly polytopes.** A polytope $P$ is $k$-neighborly if every subset of $k$ vertices of $P$ is the set of vertices of a proper face of $P$.

Proofs of the following two theorems may be found in Chapter 7 of [3].

**Theorem D.** If $P$ is a $k$-neighborly $d$-polytope then:
(i) All $k$ vertices of $P$ are affinely independent and $P$ is $k'$ neighborly for $1 < k' < k$.
(ii) if $k > [\frac{1}{2} d]$ then $P$ is a simplex.

**Theorem E.** For every $d$ and $v > d$ there exist $d$-polytopes with $v$ vertices which are $[\frac{1}{2} d]$ neighborly.
3. Proof of Theorem A. In order to prove Theorem A it is sufficient to prove

**Theorem F.** Let \( n \) and \( t \) be nonnegative integers. Let \( \mathcal{A} \) be a family of \( n + t \) convex cones with apex 0 in \( \mathbb{R}^n \) such that

\[
\bigcap \mathcal{A} = \{0\}
\]

and

\[
\bigcap \mathcal{B} \neq \{0\} \quad \text{for any } n \text{ membered subfamily } \mathcal{B} \text{ of } \mathcal{A}.
\]

Then there exists an \( n + \lfloor t/2 \rfloor \) membered subfamily \( \mathcal{C} \) of \( \mathcal{A} \) such that \( \bigcap \mathcal{C} \neq \{0\} \).

Moreover, \( n + \lfloor t/2 \rfloor \) in the last statement cannot be replaced by \( n + \lceil t/2 \rceil + 1 \).

**Proof of Theorem F.** The proof is by induction on \( n \). The cases \( n = 0 \) and \( t < 1 \) are trivial, so assume that \( n > 0 \) and that \( t > 2 \).

Let \( \mathcal{A} \) be a family of convex cones with apex 0 on \( \mathbb{R}^n \) such that \( |\mathcal{A}| = n + t \) and assume that (2) and (3) are satisfied.

**Suppose that \( A \) is not a N.D.F.** let \( \mathcal{M} \) be a maximal subset of \( \mathcal{A} \) such that

\[
|\mathcal{B}| = \dim \bigcap \mathcal{B} < n - |\mathcal{B}|.
\]

Clearly \( \phi \) satisfies (4) since \( \dim \bigcap \phi = n = n - |\phi| \).

It is not difficult to verify that \( \dim \bigcap \mathcal{M} = n - |\mathcal{M}| \) and that \( \mathcal{M} \neq \phi \) since \( \mathcal{A} \) is not a N.D.F.

Define a family \( \mathcal{A}' = \{ \bigcap \mathcal{M} \cap \{A\} : A \in \mathcal{A} \setminus \mathcal{M} \} \) of convex cones with apex 0 in \( \mathbb{R}^n = \text{span} \bigcap \mathcal{M} \).

Since \( |\mathcal{A}'| = n + t - |\mathcal{M}| = n' + t \) and since \( \mathcal{A}' \) satisfies (2) and (3) we can use the induction hypothesis to obtain a \( \mathcal{C}' \subset \mathcal{A} \setminus \mathcal{M} \) such that \( |\mathcal{C}'| > n' + \lfloor t/2 \rfloor \) and \( \bigcap \mathcal{C}' \neq \{0\} \).

Define \( \mathcal{C} = \mathcal{C}' \cup \mathcal{M} \). The subfamily \( \mathcal{C} \) has the desired properties since

\[
|\mathcal{C}| = |\mathcal{C}'| + |\mathcal{M}| > n' + \lfloor t/2 \rfloor + |\mathcal{M}| = n + \lfloor t/2 \rfloor
\]

and

\[
\bigcap \mathcal{C} = \bigcap \mathcal{C}' \neq \{0\}.
\]

**Suppose that \( A \) is a N.D.F.** Assume by contradiction that \( \bigcap \mathcal{B} = \{0\} \) for any \( n + \lfloor t/2 \rfloor \) membered subfamily \( \mathcal{B} \) of \( \mathcal{A} \). Thus for any \( \mathcal{C} \subset \mathcal{A} \) such that

\[
|\mathcal{C}| \leq n + t - (n + \lfloor t/2 \rfloor) = t - \lceil t/2 \rceil,
\]

\( \mathcal{C} \) is a \( |\mathcal{C}| \)-face of \( \mathcal{A} \).

Let \( P \) be the \((t - 1)\)-polytope described in Theorem C. It follows from (1) of Theorem C that \( P \) is a \( t - \lfloor t/2 \rfloor \) neighborholy polytope with \( n + t \) vertices. Since \( t - \lfloor t/2 \rfloor > \lfloor (t - 1)/2 \rfloor \), \( P \) is by Theorem D a \((t - 1)\)-simplex and therefore \( n + t = (t - 1) + 1 \), so that \( n = 0 \), a contradiction. This completes the proof of the first part of Theorem F.

To complete the proof of Theorem F let \( P \) be a \( \lfloor (t - 1)/2 \rfloor \) neighborholy \((t - 1)\)-polytope with \( n + t \) vertices (see Theorem E).
Let $\mathcal{A}$ be the N.D.F. described in (ii) of Theorem C. The family satisfies (2) and (3) since any N.D.F. in $\mathbb{R}^n$ satisfies (2) and also (3) if $n > 0$.

For any $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \geq n + \lceil t/2 \rceil + 1$ we have that

$$|\mathcal{A} \setminus \mathcal{B}| < t - \lfloor t/2 \rfloor - 1 < \lceil (t - 1)/2 \rceil.$$

By Theorem C(i), for any such $\mathcal{B}$, $\mathcal{A} \setminus \mathcal{B}$ is a $|\mathcal{A} \setminus \mathcal{B}|$-face of $\mathcal{A}$.

Therefore $\dim \bigcap \mathcal{B} = \dim \bigcap (\mathcal{A} \setminus (\mathcal{A} \setminus \mathcal{B})) = |\mathcal{A} \setminus \mathcal{B}| - |\mathcal{A} \setminus \mathcal{B}| = 0$ so that $\bigcap \mathcal{B} = \{0\}$. The proof of Theorem F is now complete.

REFERENCES


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