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THE INDEX OF A HOLOMORPHIC MAPPING AND THE INDEX THEOREM

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ABSTRACT. The index theorem for a harmonic mapping of riemannian manifolds is given. Let f: $M \to N$ be a holomorphic mapping of Kaehler manifolds. Then it is shown that the index of f is zero and that a Jacobi field along f is a holomorphic section of the bundle $f^* T(N)$ induced by f.

1. **Introduction**. Let M be an m-dimensional smooth compact riemannian manifold with or without boundary and N an n-dimensional smooth riemannian manifold. We take M to be oriented. Let f: $M \to N$ be a smooth mapping. In §2 we will discuss simply the variational problem of the energy integral $E(f) = \frac{1}{2} \int_M \|f_*\|^2 * 1$. In particular, we describe the second variational formula for a harmonic mapping of M to N, which was given by E. Mazet [4] and R. T. Smith [7] when $\partial M = \varnothing$. Applying Smale's index theorem [6], J. Simons got the index theorem for minimal varieties [5]. Similarly, we can get the index theorem for a harmonic mapping.

Let f: $M \to N$ be a holomorphic mapping of Kaehler manifolds. Then it is a harmonic mapping. In §3 we will compute the second variation in the complex case and prove the index of f is zero. Denote by $f^* T(N)$ the bundle induced by f from the tangent bundle $T(N)$. A section of $f^* T(N)$ is a Jacobi field along f iff it is holomorphic.

2. **The index theorem.** Let $ds_M^2 = \Sigma \omega_i^2$ (resp., $ds_N^2 = \Sigma \omega_a^{*2}$) be a riemannian metric on M (resp., N), where ω_i (resp., ω_a^*) are local differential forms in M (resp., N). (Throughout the paper, indices i, j, k (resp., a, b, c) have the range $1, \ldots, m$ (resp., $1, \ldots, n$).) We will follow the notation and terminology of [1] in this section. The structure equations of M are given by

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \quad d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum R^i_{jkl} \omega_k \wedge \omega_l.$$

Similar equations remain valid in N and we will denote the corresponding quantities by the same notations with asterisks. Let X be a vector field on M with components X^i. Then div $X = \Sigma X^i_i$, where X^i_j are the components of the covariant differential of X given by $\Sigma X^i_j \omega_j = dX^i + \Sigma X^j \omega_{ji}$. Let $* 1$ be the canonical riemannian volume form on M and ω the canonical riemannian

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volume form on the boundary ∂M. Now, Green's theorem is described as follows.

$$\int_M \sum X_i^i * 1 = \int_{\partial M} \sum X^i \nu^i \omega,$$

where ν^i are components of the unit outward normal vector field to the boundary ∂M.

Let f: $M \to N$ be a smooth mapping. Then its differential f_* is represented locally as $f^*(\omega_a^*) = \sum A_i^a \omega_i$. Denote by A_{ij}^a the components of the covariant differential of f_*, that is,

$$\sum A_{ij}^a \omega_j = dA_i^a + \sum A_j^a \omega_{ji} + \sum A_a^b \omega_{ba}^*.$$

The energy integral of f is given by

$$E(f) = \frac{1}{2} \int_M \sum (A_i^a)^2 * 1.$$

Let $\{f_t\}$ be a variation of f, i.e., $f_t = f(\cdot, t)$: $M \times I \to N$ is a smooth mapping with $f_0 = f$, where $I = [0, 1]$. If $f_t = f$ on the boundary ∂M, it is called a variation fixed on the boundary. In this case the smooth section $U = (\partial f/\partial t)_0$ of the induced bundle $f^* T(N)$ vanishes on ∂M. Let U^a be the components of U. Using Green's theorem we get

PROPOSITION 1.

$$\left(\frac{\partial E(f_t)}{\partial t}\right)_0 = -\int_M \sum U^a A_{ii}^a * 1 + \int_{\partial M} \sum U^a A_i^a \nu^i \omega.$$

f is a critical point of E for variations fixed on the boundary if it is harmonic, i.e., $\sum A_{ii}^a = 0$. Moreover, it is a critical point for all variations if it is harmonic with boundary condition $\sum A_i^a \nu^i = 0$ on ∂M.

Let $f_{t,s}$ be a 2-parameter variation of a harmonic mapping f: $M \to N$. Put $U = (\partial f_{t,s}/\partial t)_{0,0}$ and $V = (\partial f_{t,s}/\partial s)_{0,0}$. U_i^a (resp., U_{ij}^a) are the components of the first (resp., second) covariant differential of U. R. T. Smith defined the second variational operator J_f by

$$(J_f U)^a = -\left(\sum U_{ii}^a + \sum R^{*a}_{bcd} A_i^b U^c A_i^d\right),$$

where R^{*a}_{bcd} is the curvature tensor of N. Now we have

PROPOSITION 2.

$$\left(\frac{\partial^2 E(f_t, s)}{\partial t \partial s}\right)_{0,0} = \int_M \sum (J_f U)^a V^a * 1 + \int_{\partial M} \sum V^a U_i^a \nu^i \omega.$$

In the case $\partial M = \emptyset$, the above formula was given by E. Mazet [4] and R. T. Smith [7]. Later on in this section we treat only a variation fixed on the boundary. Therefore we assume that the domain of the operator J_f is $C_0^\infty(f^* T(N))$, the space of sections of $f^* T(N)$ vanishing on ∂M. An element of Ker J_f is called a *Jacobi field along f*. Let index(f) be the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues of J_f. Nullity(f) is the dimension of Ker J_f.

We will apply Smale's index theorem to our case. We refer to [5] and [6] for the basic facts about the theorem. Let M be a compact manifold with boundary. Let $g_t = g(\cdot, t)$: $M \times I \to M$ be a smooth mapping with $g_0 =$ identity. M_t denotes the images of M under g_t. $\{g_t\}$ is called a contraction of M if $M_t \subsetneq M_s$ whenever $t > s$. $\{g_t\}$ is said to be of ε-type ($\varepsilon > 0$) if the volume of M_t with respect to the induced riemannian metric is less than ε for every sufficiently large t. For every $\varepsilon > 0$, there exists a contraction of ε-type. Let f: $M \to N$ be harmonic. If Nullity($f|_{M_t}$) > 0, ∂M_t is called a conjugate boundary of f.

THEOREM 1. *Let M be a compact manifold with boundary. Let f: $M \to N$ be a harmonic mapping. Then there exists ε such that if $\{g_t\}$ is a contraction of ε-type, the number of conjugate boundaries is finite and*

$$\operatorname{index}(f) = \sum_{0<t<1} \operatorname{Nullity}(f|_{M_t}).$$

PROOF. Take normal coordinate systems $\{x^i\}$ and $\{y^a\}$ in neighborhoods of an arbitrary point p of M and $f(p)$, respectively. Let U be any section of $f^*T(N)$ with components U^a. Then at p J_f has the expression

$$(J_fU)^a = -\sum \frac{\partial^2 U^a}{\partial x_i^2} - \sum R^{*a}_{bcd} A_i^b U^c A_i^d$$

with respect to $\{x^i\}$ and $\{y^a\}$. Using the same argument as in the proof of Proposition 1.2.3 in [5], one can prove that J_f is strongly elliptic and has uniqueness in the Cauchy problem. Therefore, the result follows from the main theorem in [6].

3. **Jacobi fields along a holomorphic mapping.** Let M and N be Kaehler manifolds with dimensions $2m$ and $2n$, respectively. The hermitian metric of M (resp., N) can be written as $ds_M^2 = \sum \omega_i \bar{\omega}_i$ (resp., $ds_N^2 = \sum \omega_a^* \bar{\omega}_a^*$) where ω_i (resp., ω_a^*) are local complex-valued linear differential forms of type (1, 0) in M (resp., N). The connection forms ω_{ij} in M are characterized by the conditions

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \quad \omega_{ij} + \bar{\omega}_{ji} = 0,$$

$$d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \sum R^j_{ikl} \omega_k \wedge \bar{\omega}_l.$$

Similar equations are valid in N. We will denote the corresponding quantities by the same notations with asterisks.

Assume that M is a compact Kaehler manifold with or without boundary. Let f: $M \to N$ be a smooth mapping. Then we can put

$$f^*(\omega_a^*) = \sum A_i^a \omega_i + \sum A_{i^*}^a \bar{\omega}_i = \sum A_I^a \omega_I,$$

where we use the notations $i^* = m + 1$ and $\omega_{i^*} = \bar{\omega}_i$. (In this section, indices

I, J, K run from 1 to $2m$.) Set

$$E'(f) = \int_M \sum (A_i^a)^2 * 1 \quad \text{and} \quad E''(f) = \int_M \sum (A_{i^*}^a)^2 * 1.$$

Then the energy integral is given by $E(f) = E'(f) + E''(f)$. Let A_{IJ}^a be the components of the covariant differential of f_*. From the definition f is harmonic iff $\sum A_{ii^*}^a + \sum A_{i^*i}^a = 0$. But since A_{IJ}^a is symmetric in I and J, it is harmonic if $\sum A_{ii^*}^a = 0$ or if $\sum A_{i^*i}^a = 0$. Now f is holomorphic (resp., antiholomorphic) iff $A_{i^*}^a = 0i$ (resp., $A_i^a = 0$). Hence if it is holomorphic or antiholomorphic, it is harmonic.

Let f_t ($t \in I$) be a variation fixed on the boundary. Then we know

PROPOSITION 3 (A. LICHNEROWICZ). *For all* $t \in I$, $dE'(f_t)/dt = dE''(f_t)/dt$.

Next, we want to study the second variation of E'. Let $f_{t,s}$ be a 2-parameter variation of a harmonic mapping $f\colon M \to N$, i.e., $f_{t,s} = f(\cdot, t, s)\colon M \times I \times I \to N$ is a smooth mapping with $f_{0,0} = f$ and $f_{t,s} = f$ on ∂M. The differential $(f_{t,s})_*$ is regarded as a $f_{t,s}^* T(N)$-valued 1-form on $M \times I \times I$ and it has the local expression

$$f_{t,s}^*(\omega_a^*) = \sum B_I^a \omega_I + B_0^a dt + B_{-1}^a ds = \sum B_\Phi^a \omega_\Phi, \tag{1}$$

where $\omega_0 = dt$, $\omega_{-1} = ds$. (In this section indices Φ, Ψ, Θ have the range $-1, 0, 1, \ldots, 2m$.) There is a covariant differential operator naturally induced on the bundle $f_{t,s}^* T(N)$. The covariant differential of $(f_{t,s})_*$ is given by

$$\sum B_{\Phi\Psi}^a \omega_\Psi = dB_\Phi^a - \sum B_\Psi^a \omega_{\Phi\Psi} + \sum B_\Phi^a \omega_{ba}, \tag{2}$$

where $\omega_{\Phi 0} = \omega_{0\Phi} = \omega_{\Phi(-1)} = \omega_{(-1)\Phi} = 0$ and $B_{\Phi\Psi}^a$ is symmetric in Φ, Ψ. Note that $(B_i^a)_{0,0} = A_i^a$, $(B_{ij}^a)_{0,0} = A_{ij}^a$. In addition, $(B_0^a)_{0,0}$, $(B_{-1}^a)_{0,0}$ are components of $U = (\partial f_{t,s}/\partial t)_{0,0}$ and $V = (\partial f_{t,s}/\partial s)_{0,0}$, respectively.

LEMMA 1. *Let $B_{\Phi\Psi\Theta}^a$ be the components of the second covariant differential of $(f_{t,s})_*$. Then we have*

$$B_{\Phi\Psi\Theta}^a - B_{\Phi\Theta\Psi}^a = \sum B_h^a \delta_\Phi^i R_{ijk}^h \left(\delta_\Psi^j \delta_\Theta^{k^*} - \delta_\Theta^j \delta_\Psi^{k^*}\right)$$
$$- \sum B_\Phi^b R^{*a}{}_{bcd}\left(B_\Psi^c \overline{B_{\Theta^*}^d} - B_\Theta^c \overline{B_{\Psi^*}^d}\right),$$

where $0^* = 0$, $(-1)^* = -1$, $i^* = m + i$ *and* $(m + i)^* = i$.

PROOF. Taking the exterior derivative of (2) and using the structure equations in M and N, we get the result.

PROPOSITION 4.

$$\left(\frac{\partial^2 E'}{\partial t \partial s}\right)_{0,0} = \left(\frac{\partial^2 E''}{\partial t \partial s}\right)_{0,0} = \int_M (U, J_f' V) * 1$$
$$= \int_M \sum \left(U^a \overline{(J_f' V)^a} + \overline{U^a} (J_f' V)^a\right) * 1,$$

where the differential operator J_f': $C_0^\infty(f^*T(N)) \to C^\infty(f^*T(N))$ *is defined to be*

$$(J_f'V)^b = -\sum V^b_{ii^*} - \sum A_i^a R^{*b}_{acd}\left(\overline{A_i^d}\, V^c - A^c_{i^*}\overline{V^d}\right).$$

PROOF. From Proposition 3 it follows that $\partial^2 E'/\partial t\partial s = \partial^2 E''/\partial t \partial s$. Hence we will calculate only $\partial^2 E'/\partial t\partial s$. For any fixed $(u, v) \in I \times I$, let i_{uv} be the injection of M to $M \times I \times I$ such that $i_{uv}(p) = (u, v, p)$ for $p \in M$. Denote by T_{uv} the bundle induced by $f_{t,s} \circ i_{uv}$ from $T(N)$. Then $(\partial f_{t,s}/\partial t)_{u,v}$ and $(\partial f_{t,s}/\partial s)_{u,v}$ are smooth sections of T_{uv} with components $(B_0^a)_{u,v}$ and $(B^a_{-1})_{u,v}$, respectively. The differential of $f_{t,s} \circ i_{uv}$ is a T_{uv}-valued 1-form with components B_I^a in the left-hand side of (1). Hence we can put

$$E'(f_{t,s}) = E'(f_{t,s} \circ i_{uv})$$
$$= \int_M \sum |B_i^a|^2 * 1.$$

Since

$$\sum B_i^a \overline{\partial B_i^a/\partial t} + \sum \overline{B_i^a}\, \partial B_i^a/\partial t = \sum B_i^a \overline{B_{i0}^a} + \sum \overline{B_i^a}\, B_{i0}^a,$$

we get

$$\partial E'/\partial t = \int_M \sum \left(B_i^a \overline{B_{i0}^a} + \overline{B_i^a}\, B_{i0}^a\right) * 1.$$

Moreover, we have

$$\frac{\partial^2 E'}{\partial t \partial s} = \int_M \sum \left(B_{i0}^a \overline{B_{i(-1)}^a} + \overline{B_{i0}^a}\, B_{i(-1)}^a + B_i^a \overline{B_{i0(-1)}^a} + \overline{B_i^a}\, B_{i0(-1)}^a\right) * 1.$$

Using Lemma 1 and Green's theorem, we obtain the desired formula.

LEMMA 2. J_f' *is equal to the operator* J_f'' *given by*

$$(J_f''V)^b = -\sum V^b_{i^*i} - \sum A^a_{i^*} R^{*b}_{acd}\left(\overline{A^d_{i^*}}\, V^c - A_i^c \overline{V^d}\right).$$

PROOF. If the second variation of E'' is calculated by the method in the proof of Proposition 4, it follows that $\partial^2 E''/\partial t \partial s = \int_M (U, J_f'' V) * 1$. Hence we get $\int_M (U, J_f' V) * 1 = \int_M (U, J_f'' V) * 1$. This is true for every variation. Thus we obtain $J_f' = J_f''$.

From Proposition 4 and Lemma 2 it is evident that a section U of the induced bundle is a Jacobi field iff $J_f' U = 0$ or $J_f'' U = 0$.

Let f: $M \to N$ be a holomorphic mapping. A section $U \in C^\infty(f^*T(N))$ is said to be holomorphic if $U^a_{i^*} = 0$. This condition implies the following: For any point $p \in M$, take local complex coordinate systems $\{z_i\}$ and $\{w_a\}$ in the neighborhoods of p and $f(p)$, respectively. U has the local expression $U = \sum U^a \partial/\partial w_a + \sum \overline{U^a}\partial/\partial \overline{w}_a$. Then it is holomorphic iff U^a are always holomorphic functions of the z_i's.

THEOREM 2. *Let* f: $M \to N$ *be a holomorphic mapping. Then the index of* f *is zero. A section of* $f^*T(N)$ *vanishing on* ∂M *is a Jacobi field along* F *iff it is* *holomorphic. Hence the nullity of f is equal to the dimension of the space of holomorphic sections vanishing on ∂M.*

PROOF. Let

$$I(U, V) = \int_M (U, J_f' V) * 1 = \int_M (V, J_f' U) * 1.$$

Then it is a symmetric form on $C_0^\infty f^* T(N)$. Since f is holomorphic, $A_{i^*}^a = 0$. Hence $(J_f' U)^b = (J_f'' U)^b = -\sum U_{i^*i}^b$. Applying the divergence theorem, we get

$$I(U, U) = -\int_M \sum (U^a \overline{U_{i^*i}^a} + \overline{U^a} U_{i^*i}^a) * 1 = 2\int_M \sum |U_{i^*}^a|^2 * 1.$$

Thus the index of f is zero. U is a Jacobi field iff $U_{i^*}^a = 0$.

For an antiholomorphic mapping, we can obtain a similar result. From the above theorem, it easily follows that a vector field on a Kaehler manifold M is a Jacobi field along the identity mapping of M iff it is holomorphic. For example, let M be a compact Kaehler manifold with negative first Chern class. Then the Nullity of the identity mapping of M is zero.

REFERENCES

1. S. I. Goldberg and T. Ishihara, *Harmonic quasiconformal mappings of riemannian manifolds*, Bull. Amer. Math. Soc. **80** (1974), 562–566.

2. S. I. Goldberg, T. Ishihara and N. C. Petridis, *Mappings of bounded dilatation of riemannian manifolds*, J. Differential Geometry **10** (1975), 619–630.

3. A. Lichnerowicz, *Applications harmoniques et variétés kahleriennes*, Rend. Sem. Mat. Fis. Milano **39** (1969), 1–12.

4. E. Mazet, *La formule de la variation seconde de l'energie au voisinage d'une application harmonique*, J. Differential Geometry **8** (1973), 279–296.

5. J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.

6. S. Smale, *On the Morse index theorem*, J. Math. Mech. **14** (1965), 1049–1056.

7. R. T. Smith, *The second variation formula for harmonic mappings*, Proc. Amer. Math. Soc. **47** (1975), 229–236.

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