STICKY ARCS IN $E^n$ ($n > 4$)

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Abstract. Let $A$ and $B$ be arcs in $E^3$, Euclidean 3-space. Then $A$ can be "slipped" off $B$; i.e., there exists a homeomorphism of $E^3$ onto itself, arbitrarily close to the identity, such that $h(A) \cap B = \emptyset$. The purpose of this note is to show that arcs in $E^n$ ($n > 4$) do not always enjoy this property. The examples depend heavily on a recent result of McMillan.

1. Introduction. If $X$ and $Y$ are subsets of $E^n$, Euclidean $n$-space, we say that $X$ can be slipped off $Y$ in $E^n$ if for each $\epsilon > 0$ there is an $\epsilon$-homeomorphism $h: E^n \to E^n$ such that $h(X) \cap Y = \emptyset$; otherwise, we say $X$ cannot be slipped off $Y$. Results of Armentrout [1] and McMillan [5] show that if $A$ and $B$ are arcs in $E^3$, then $A$ can be slipped off $B$. We show that this is false in higher dimensions by proving the following

Theorem. There exist cellular arcs $A$ and $B$ in $E^n$ ($n > 4$) such that $A$ cannot be slipped off $B$.

Existence of $A$ and $B$. McMillan [4] has shown, for each $n > 4$, there exists an arc $C$ in an $n$-manifold $M^n$ which has no neighborhood in $M^n$ which embeds in $E^n$. Without loss of generality, we may assume that $C$ is the union of arcs $A'$ and $B'$ whose intersection is a common endpoint and such that $A'$ and $B'$ have neighborhoods $U$ and $V$, respectively, which embed in $E^n$. Let $f: U \to E^n$ and $g: V \to E^n$ be embeddings. By [2] and a correct choice of $U$ and $V$, we may assume that $f$ and $g$ agree on $U \cap V$. We now let $A = f(A')$ and $B = g(B')$.

Proof of Theorem. Let $B_1$ and $B_2$ be concentric $n$-balls centered at $f(A' \cap B')$ such that $B_1 \subset \text{Int } B_2 \subset f(U \cap V) = g(U \cap V)$. We suppose that $A$ can be slipped off $B$ and choose a homeomorphism $h: E^n \to E^n$ such that $h(A) \cap B = \emptyset$. Furthermore, $h$ is chosen so close to the identity that the following conditions are satisfied.

1. $h(\text{Bd } B_2) \cap B_1 = \emptyset$.

2. There is an embedding $e: B_2 - \text{Int } B_1 \to E^n$ such that $e|\text{Bd } B_1 = \text{identity}$ and $e|\text{Bd } B_2 = h|\text{Bd } B_2$.

3. The embedding $e$ is so close to the identity that $e(A \cap (B_2 - \text{Int } B_1)) \cap B = \emptyset$.

The existence of the embedding $e$ is a corollary of the Kirby-Edwards local contractibility theorems [3] and is used explicitly in [6].

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Consider the homeomorphism $H: E^n \to E^n$ given by $H|B_1 = \text{identity}$, $H|(B_2 - \text{Int } B_1) = e$, and $H|(E^n - B_2) = h|(E^n - B_2)$. Thus, $H \circ f: U \to E^n$ and $g: V \to E^n$ are embeddings which agree on a neighborhood of $A' \cap B'$ in $M^n$ and such that $H \circ f(A') \cap g(B')$ is a single point. By choosing even smaller neighborhoods $U'$ and $V'$ of $A'$ and $B'$, respectively, we may assume that $f' = H \circ f|U'$ and $g' = g|V'$ agree on $U' \cap V'$ and that $f'(U') \cap g'(V') = f'(U' \cap V') = g'(U' \cap V')$. Thus, $f'$ and $g'$ can be used to give an embedding of $U' \cup V'$ into $E^n$ which yields a contradiction.

An examination of McMillan's arc $C$ shows that the interior of $C$ has a neighborhood in $M^n$ which is homeomorphic with the suspension of a set $X$, $\Sigma X$, the homeomorphism sending $C$ to $\Sigma\{x\}$, $x \in X$. The set $X$ is obtained from a piecewise-linear $(n - 1)$-manifold by identifying a cell-like set in its interior to the point $x$. This neighborhood may be used to show that any proper subarc of $C$ can be pushed arbitrarily close to one of its endpoints by a homeomorphism of $M^n$ which has support in an arbitrary neighborhood in $M^n$ of the subarc. This fact implies that any subarc of $C$ is cellular. Therefore we may assume that $A$ and $B$ are cellular.

REFERENCES


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