PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 2, October 1977

CRITICAL EDGES IN SUBDIRECTLY IRREDUCIBLE LATTICES

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ABSTRACT. In studying a particular subdirectly irreducible lattice it is often important to know the location of a *critical edge*, that is, a quotient that is collapsed by every nontrivial congruence relation. We show that a critical edge can always be found in a rather natural location.

In studying a particular subdirectly irreducible lattice L, it is often important to know the location of a *critical edge* of L, that is, a quotient u/v of L that is collapsed by every nontrivial congruence relation on L. Disregarding the trivial case in which L is distributive, and is therefore just a two-element chain, we shall show that such an edge can always be found in a rather natural location.

A nondistributive lattice, of course, contains a sublattice isomorphic to the diamond M_3 or the pentagon N (see Figure 1). For distinct elements a, b and c of a lattice we write $M(a, b, c)$ if $a + b = b + c = c + a$ and $ab = bc = ca$, and we write $N(a, b, c)$ if $bc < a < c < a + b$. Call a quotient u/v of L an *M-quotient* if it is an edge of a diamond, that is, if there exist $a, b, c \in L$ with $M(a, b, c)$ such that either $u/v = (a + b)/a$ or $u/v = a/ab$; call u/v an *N-quotient* if $N(v, b, u)$ for some $b \in L$.

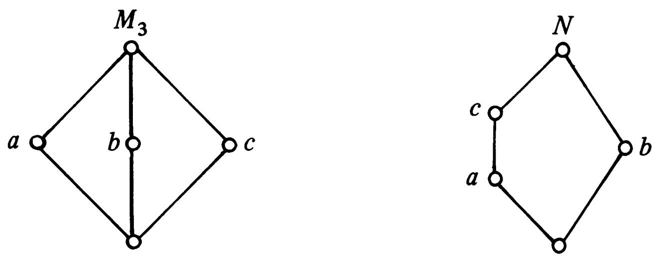

FIGURE 1

THEOREM 1. *Every nonmodular, subdirectly irreducible lattice contains a critical edge that is an N-quotient.*

Received by the editors December 16, 1976.
AMS (MOS) subject classifications (1970). Primary 06A20; Secondary 06A30.
Key words and phrases. Subdirectly irreducible lattice, critical edge.
[1]The work of the first author was supported by NSF Grant #MCS 76-06447; the work of the second author was supported by National Research Council Operating Grants 69-0489 and 69-0557.

THEOREM 2. *Every modular, nondistributive, subdirectly irreducible lattice contains a critical edge that is an M-quotient.*

PROOF OF THEOREM 1. Let L be a nonmodular, subdirectly irreducible lattice; then L has both an N-quotient and a critical edge. We want to show that L has an N-quotient that is also a critical edge. For any N-quotient a/b and any critical edge c/d there exist a natural number n and quotients $a/b = x_0/y_0, x_1/y_1, \ldots, x_n/y_n$ with $d \leqslant y_n < x_n \leqslant c$ such that for each $i < n$, x_{i+1}/y_{i+1} is a subquotient of a transpose of x_i/y_i. Our objective is to show that for suitable a/b and c/d, n can be taken to be 0, for this means that some N-quotient a/b is a subquotient of a critical edge c/d, and is therefore itself a critical edge.

Let us suppose, to the contrary, that n is never 0, and choose a/b and c/d, as well as the quotients x_i/y_i so that n is as small as possible. We may assume that x_n/y_n is a subquotient of a lower transpose u/v of x_{n-1}/y_{n-1}, that is, $x_{n-1} = y_{n-1} + u$, $v = y_{n-1}u$ and $v \leqslant y_n < x_n \leqslant u$. From the minimality of n it follows that no subquotient of x_{n-1}/y_{n-1} is a critical edge, and hence a nontrivial subquotient of x_{n-1}/y_{n-1} cannot be a transpose of a subquotient of x_n/y_n. Thus the quotients $(y_{n-1} + x_n)/(y_{n-1} + y_n)$ and $x_n/(y_{n-1} + y_n)x_n$, being transposes of each other, must be trivial, that is, we must have $x_n \leqslant y_{n-1} + y_n$. However, this implies that $N(y_n, y_{n-1}, x_n)$, and x_n/y_n is thus both a critical edge and an N-quotient. This contradiction completes the proof.

PROOF OF THEOREM 2. Let L be a modular, nondistributive, subdirectly irreducible lattice. Then any two nontrivial quotients in L have nontrivial subquotients that are projective to each other. Hence every nontrivial quotient has a subquotient that is a critical edge, and it therefore suffices to show that every nontrivial subquotient of an M-quotient is projective to an M-quotient.

Let $M(a, b, c)$ and $c \leqslant y < x \leqslant a + b$. Set $a' = ax$, $b' = bx$, $c' = (a' + b')c$ and $y' = (a' + b')y$. We then have $M(a', b', c')$, and $(ax + bx)/y'$ is a lower transpose of x/y. Now put $a'' = a' + b'y'$, $b'' = b' + a'y'$ and $c'' = y'$. We have $M(a'', b'', c'')$, and the transpose $(a'' + b'')/c'' = (a' + b')/y'$ of x/y is therefore an M-quotient.

In [**1**], Dilworth and Freese show that any lattice L can be embedded in a strongly atomic lattice L' such that L and L' generate the same variety. The two corollaries that follow were inspired by this fact, although we actually use only a weaker and more obvious result.

COROLLARY 3. *Every completely join-irreducible, nonmodular variety of lattices is generated by a subdirectly irreducible lattice having a critical edge that is both an N-quotient and a prime quotient.*

COROLLARY 4. *Every completely join-irreducible, modular, nondistributive variety of lattices is generated by a subdirectly irreducible lattice having a critical edge that is both an M-quotient and a prime quotient.*

PROOF OF COROLLARY 3. Let $\mathcal{V}$ be a completely join-irreducible, nonmodular variety of lattices. Then $\mathcal{V}$ is generated by a subdirectly irreducible lattice L. Of course, L is nonmodular, and by Theorem 1 it therefore has a critical edge a/b that is an N-quotient. Let L^σ be the lattice of all ideals of L. We shall consider L as a sublattice of L^σ in the obvious manner. Since a is compact in L^σ, there exists $c \in L^\sigma$ with $b \leqslant c \prec a$. Obviously a/c is an N-quotient. Let θ be a maximal congruence relation on L^σ that does not identify a and c. The lattice $L' = L^\sigma/\theta$ is in $\mathcal{V}$, and the homomorphism $x \to x'$ of L^σ onto L' is one-to-one on L since it does not identify a and b. Hence L' generates $\mathcal{V}$. Furthermore, the critical edge a'/c' is both an N-quotient and a prime quotient.

The proof of Corollary 4 is similar.

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