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SEMICONTINUOUS AND IRRESOLUTE IMAGES OF S-CLOSED SPACES

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ABSTRACT. A topological space is S-closed if and only if every semi-open cover of X has a finite subcollection whose closures cover X. The images of S-closed spaces under various mappings are investigated culminating in this main result: A Hausdorff space X is S-closed if and only if the irresolute image of X in any Hausdorff space is closed.

1. **Introduction.** In [8] the concept of an S-closed space was defined. In this paper, a characterization of Hausdorff S-closed spaces are given using a certain class of functions. No separation axioms are assumed unless otherwise specified.

2. **Preliminaries.** In order for this note to be as self-contained as possible, the following basic definitions are given. A subset V of a topological space is semi-open if and only if $V^\circ \subset V \subset \overline{V^\circ}$. A topological space X is S-closed if and only if every semi-open cover of X has a finite subcollection whose closures cover X. A subset F is semiclosed if its complement is semi-open. A function f: $X \to Y$ is said to be irresolute (semicontinuous) if the inverse image of every semi-open (open) set is semi-open. A topological space is extremally disconnected if the closure of every open set is open. If A is any subset of a topological space X, then the semiclosure of A, denoted $\underline{A}$, is the intersection of all semi-closed sets in X that contain A. The usual closure of a set A will be denoted by $\overline{A}$ and its interior by A°.

3. **Main results.**

THEOREM 3.1. *A function f: $X \to Y$ is semicontinuous if and only if for every subset A of $f(\underline{A}) \subset \overline{f(A)}$* [1, *Theorem* 1.16].

THEOREM 3.2. *The semicontinuous surjection of an S-closed space onto any Hausdorff space is H-closed.*

PROOF. Let f: $X \to Y$ be a semicontinuous surjection and $\{V_a\}$ an aribtrary open cover of Y. Then $\{f^{-1}(V_a)\}$ is a semi-open cover of X. By hypothesis, there exists a finite subfamily such that $\overline{\bigcup_1^n f^{-1}(V_{a_i})} = X$. Notice that $\bigcup_1^n f^{-1}(V_{a_i})$ being dense in X implies $\underline{\bigcup_1^n f^{-1}(V_{a_i})} = X$. By Theorem 3.1,

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$$Y = f(X) = f\left[\underline{\bigcup_1^n f^{-1}(V_{a_i})}\right] \subset \overline{f\left[\bigcup_1^n f^{-1}(V_{a_i})\right]}$$

$$= \overline{\bigcup_1^n V_{a_i}} = \bigcup_1^n \overline{V_{a_i}}.$$

Therefore, Y is H-closed.

The proof of Theorem 3.2 remains valid without the assumption of Y being Hausdorff; that is, Y enjoys the "H-closed covering property."

COROLLARY 3.3. *The semicontinuous surjection of an S-closed space onto any regular space is compact.*

THEOREM 3.4. *A function f: $X \to Y$ is irresolute if and only if for every subset A of X, $f(\underline{A}) \subset \underline{f(A)}$* [3, *Theorem* 1.5].

THEOREM 3.5. *If f: $X \to Y$ is an irresolute surjection from an S-closed space X, then Y is S-closed.*

PROOF. Let $\{V_a\}$ be a semi-open cover of Y. Then $\{f^{-1}(V_a)\}$ is a semi-open cover of X and, by hypothesis, has a finite subfamily such that $\overline{\bigcup_1^n f^{-1}(V_{a_i})} = X$. Since $\bigcup_1^n f^{-1}(V_{a_i})$ is dense in X, $\underline{\bigcup_1^n f^{-1}(V_{a_i})} = X$. Therefore, by Theorem 3.5,

$$Y = f(X) = f\left[\underline{\bigcup_1^n f^{-1}(V_{a_i})}\right] \subset \underline{f\left[\bigcup_1^n f^{-1}(V_{a_i})\right]}$$

$$= \underline{\bigcup_1^n V_{a_i}} \subset \bigcup_1^n \overline{V_{a_i}}.$$

Hence, Y is S-closed.

COROLLARY 3.6. *S-closed is a semitopological property, and hence a topological property* [3, *Theorem* 1.15].

We note that the S-closed property is not, in general, preserved by continuous functions: βR is the continuous image of βN, but βN is S-closed [8, Theorem 5] and βR is not.

The following lemma is well known and will be stated without proof.

LEMMA 3.7. *A topological space X is extremally disconnected if and only if every two disjoint open sets in X have disjoint closures.*

DEFINITION 3.8. A filterbase F is said to s-accumulate to a point x if for every semi-open set V containing x and for every $F_a \in F$, $F_a \cap \overline{V} \neq \varnothing$.

In the following theorem we use the fact that an S-closed Hausdorff space is extremally disconnected [8, Theorem 7].

THEOREM 3.9. *A space is S-closed if and only if every filterbase has an s-accumulation point* [8, *Theorem* 2].

THEOREM 3.10. *The irresolute image of any S-closed Hausdorff space in any Hausdorff space is closed.*

PROOF. Let $f: X \to Y$ be an irresolute function from an S-closed space X to a Hausdorff space Y. Let $y \in \overline{f(X)}$ and $N(y)$ be the open neighborhood filterbase about y. By hypothesis, the filterbase $F = f^{-1}[N(y)]$ has an s-accumulation point x. We claim that the filterbase $f(F)$ accumulates to $f(x)$ in the usual sense. Indeed, let V be any open set containing $f(x)$. Then $f^{-1}(V)$ is a semi-open set containing x, and therefore for every $W \in N(y), f^{-1}(W) \in F$, and $f^{-1}(W) \cap \overline{f^{-1}(V)} \neq \emptyset$. But by Lemma 3.7, we have $f^{-1}(W)^{\circ} \cap f^{-1}(V)^{\circ} \neq \emptyset$. Therefore,

$$\emptyset \neq f\left[f^{-1}(W)^{\circ} \cap f^{-1}(V)^{\circ}\right] \subset f\left[f^{-1}(W) \cap f^{-1}(V)\right] \subset W \cap V.$$

Since W and V were arbitrarily chosen, we have that $f(F)$ accumulates to $f(x)$ in the usual sense. But $f(F)$ is a finer filterbase than $N(y)$, hence $N(y)$ accumulates to $f(x)$. Since $N(y)$ obviously converges to y, we have by the Hausdorff property that $f(x) = y$. Hence, $y \in f(X)$ and $f(X)$ is closed in Y.

THEOREM 3.11. *If every irresolute image of a Hausdorff space X in any Hausdorff space Y is closed, then X is S-closed.*

PROOF. Suppose that X is not S-closed. Then by Theorem 3.9 there exists a filterbase F with no s-accumulation point. This implies that for every $x \in X$, there exist an open set $V(x)$ about x and an element $F_{a(x)}$ of F such that $F_{a(x)} \cap \overline{V(x)} = \emptyset$. Let $S = \{$all finite intersections of sets of the form $(X - \overline{V(x)})\}$. It is evident that S forms an open filterbase. Select an object ∞ not in X and consider the space $\hat{X} = X \cup \{\infty\}$ with the following topology: neighborhoods of points in X are unchanged, and a basic neighborhood system of ∞ is $N(\infty) = \{S_b \cup \{\infty\} | S_b \in S\}$. It is easy to verify that $\hat{X}$ is Hausdorff. Considering the inclusion map $i: X \to \hat{X}$, we see that i is irresolute and that $i(X)$ is not closed in $\hat{X}$. Therefore, the theorem follows by contraposition.

Combining the results of Theorems 3.10 and 3.11 we have the following characterization.

COROLLARY 3.12. *A Hausdorff space X is S-closed if and only if the irresolute image of X in any Hausdorff space is closed.*

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