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LOCALLY FREE ACTIONS AND STIEFEL-WHITNEY NUMBERS. II

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ABSTRACT. This paper determines the possible bordism classes of manifolds with a locally free G action for G one of $S^1 \times S^1$, $(S^1)^4$ or S^3 and gets partial information for $(S^1)^3$.

1. **Introduction.** An action of a k-dimensional Lie group G on an n-manifold M^n is called *locally free* if all isotropy groups are discrete subgroups of G. In [4], Winkelnkemper analyzed the cobordism classes of smooth manifolds admitting locally free $S^1 \times S^1$ actions. Unfortunately, at the bottom of page 328 of [4] there is the phrase "it follows easily", and as is to be anticipated the assertion is false.

The objective of this paper is to correct this error. For $S^1 \times S^1$ the proper result is

THEOREM. *The set of classes $\alpha \in \mathfrak{N}_n$ with $n \geqslant 2$ represented by a manifold M^n admitting a locally free $S^1 \times S^1$ action consists precisely of the classes with $w_n(\alpha) = 0$.*

Note. We restrict to $n \geqslant 2$ to avoid nonsense about G actions on the empty manifold of dimension $n < 2$.

We also determine the classes represented by manifolds with $(S^1)^4$ and S^3 actions and examine $(S^1)^3$ actions. Our Proposition 3.3 shows that the main point of [4, Proposition 2.4, p. 324] is correct. We merely illustrate with a different example.

2. **Fiberings over projective spaces.** In order to construct locally free G actions, we will use certain standard fiberings over products of projective spaces.

For ξ a vector bundle over a space X, let $RP(\xi)$ be the projective space bundle of ξ consisting of lines in the fibers of ξ and let λ be the canonical line bundle over $RP(\xi)$ with total space the pairs (α, x), with α a line in a fiber of ξ and x a vector of ξ in the line α. Denote by $\underline{n}$ the trivial n-plane bundle

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over a space, and let $RP(n) = RP(\underline{n+1})$, where $\underline{n+1}$ is the trivial bundle over a point, be projective n-space.

Let $RP(n_1, n_2, \ldots, n_k)$ be the projective space bundle of the bundle $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_k$ over $RP(n_1) \times \cdots \times RP(n_k)$, where λ_i is the pull-back of the canonical bundle over the ith factor.

We also need a manifold which is slightly more complicated to describe. Over $RP(1) \times RP(1)$ one has a projective space bundle $RP(\lambda_1 \oplus \underline{1})$, with projection π and canonical line bundle λ. Over $RP(\lambda_1 \oplus \underline{1})$ one has a projective space bundle $RP(\lambda \oplus \pi^*\lambda_2 \oplus \underline{1})$.

PROPOSITION 2.1. *There exist indecomposable elements x_n, $n \geqslant 4$, $n \neq 2^t - 1$, in $\mathfrak{N}_n$ which are represented by manifolds admitting a locally free $S^1 \times S^1$ action.*

PROOF. If $n = 4s$, $s \geqslant 1$, let x_n be the class of $RP(1, \ldots, 1, 0)$ ($2s$ ones). If $n = 4s + 2$, $s \geqslant 1$, let x_n be the class of $RP(1, \ldots, 1, 0, 0, 0)$ ($2s$ ones). If $n = 2^p(2q + 1) - 1$, with $p, q > 0$ and $2^pq = s + 3 \geqslant 4$, let x_n be the class of $RP(2^p, 3, 1, \ldots, 1, 0)$ (s ones). Finally, if $n = 5$, let x_n be the class of the manifold $RP(\lambda \oplus \pi^*\lambda_2 \oplus \underline{1})$.

For $n \neq 5$, these classes x_n are shown to be indecomposable in [2, Proposition 7.1], the same being noted in [3, page 187]. For $n = 5$, indecomposability may be verified by direct calculation. One can, however, argue as follows. As noted in [3, page 187], $RP(2, 1, 0)$ is indecomposable and fibers over $RP(2) \times RP(1)$. The manifold $RP(\lambda_1 \oplus \underline{1})$ is $K \times RP(1) = RP(1, 0) \times RP(1)$, where K is the Klein bottle, and under the map $f \times 1$: $RP(1, 0) \times RP(1) \rightarrow RP(2) \times RP(1)$ with f pulling the canonical line bundle over $RP(2)$ back to that over $RP(1, 0)$, the fibration of $RP(2, 1, 0)$ over $RP(2) \times RP(1)$ pulls back to $RP(\lambda \oplus \pi^*\lambda_2 \oplus \underline{1})$. Since $f \times 1$ is a degree one map, Proposition 2.4 of [2] applies, and the class of $RP(\lambda \oplus \pi^*\lambda_2 \oplus \underline{1})$ is decomposable.

It remains to exhibit $S^1 \times S^1$ actions on our manifolds which are locally free. For $RP(n_1, \ldots, n_k)$, one notes that $RP(n_1, \ldots, n_k)$ is a quotient space of $S^{n_1} \times \cdots \times S^{n_k} \times S^{k-1}$ by an action of $Z_2 \times \cdots \times Z_2$ ($k + 1$ times). One has the obvious free action of S^1 on S^{2j+1}, and by forming product actions, one obtains a locally free $S^1 \times \cdots \times S^1$ (m times) action on $RP(n_1, \ldots, n_k)$ if $m \leqslant$ number of n_i which are odd. Finally, $RP(\lambda \oplus \pi^*\lambda_2 \oplus \underline{1})$ is the quotient space of $S^1 \times S^1 \times S^1 \times S^2$ under the identifications induced by

$$\begin{aligned}
(z_1, z_2, z_3, (u, v, w)) &\sim (z_1, z_2, -z_3, (-u, v, w)),\\
&\sim (z_1, z_2, z_3, (-u, -v, -w)),\\
&\sim (-z_1, z_2, -\bar{z}_3, (u, v, w)),\\
&\sim (z_1, -z_2, z_3, (u, -v, w))
\end{aligned}$$

where $z_i \in C$, $\|z_i\| = 1$, $(u, v, w) \in \mathbf{R}^3$ with $u^2 + v^2 + w^2 = 1$ and the bar denotes complex conjugation. These identifications are compatible with the free action of $S^1 \times S^1$ on $S^1 \times S^1 \times S^1 \times S^2$ by

$$\big((z, z'), \quad (z_1, z_2, z_3, (u, v, w))\big) \to (z \cdot z_1, z' \cdot z_2, z_3, (u, v, w))$$

inducing a locally free action of $S^1 \times S^1$ on $RP(\lambda \oplus \pi^*\lambda_2 \oplus \underline{1})$.

Note. In the remark following Proposition 2.4 of [4] the manifold M^4 is $RP(1, 1, 0)$. This manifold does not bound, nor does $M^6 = M^4 \times RP(2)$. Thus, the remark in fact gives a counterexample to the proposition it is trying to illustrate. Recognition of this fact led to this paper.

As in [4], one has the observation that if a k-dimensional Lie group G acts locally freely on a manifold M^n then the action defines k linearly independent vector fields on M^n and hence for $i > n - k$ the Stiefel-Whitney classes w_i vanish. Combining this with the proposition gives

THEOREM 2.2. *The set of classes $\alpha \in \mathfrak{N}_n$ with $n \geqslant 2$ represented by a manifold M^n admitting a locally free $S^1 \times S^1$ action consists precisely of the classes with $w_n(\alpha) = 0$.*

PROOF. Both sets of classes are the elements of ideals in $\mathfrak{N}_* = Z_2[x_i | i \neq 2^t - 1]$ described by $x_2 = [RP(2)]$ and $w_n(x_n) = 0$ if $n > 2$. Since the classes represented by manifolds with locally free $S^1 \times S^1$ action all have even Euler characteristic, one has an inclusion and the ideals must coincide.

3. Higher dimensional groups.

PROPOSITION 3.1. *The set of classes $\alpha \in \mathfrak{N}_n$ with $n \geqslant 4$ represented by a manifold M^n admitting a locally free $(S^1)^4$ action consists precisely of the classes for which all Stiefel-Whitney numbers divisible by w_n, w_{n-1}, w_{n-2} and w_{n-3} are zero.*

PROOF. Using $(S^1)^4$ actions similar to those of §2 on the manifolds used in [2, Proposition 7.2] gives the result.

PROPOSITION 3.2. *The set of classes $\alpha \in \mathfrak{N}_n$ with $n \geqslant 3$ represented by a manifold M^n admitting a locally free S^3 action consists precisely of the classes for which all Stiefel-Whitney numbers divisible by w_n, w_{n-1}, and w_{n-2} are zero.*

PROOF. If $i = 4k + 2$, let x_i be the class of $RP(3, \ldots, 3, 0, 0, 0)$ (k threes). If $i = 4k \geqslant 8$, let x_i be the class of $RP(3, 3, 0, \ldots, 0)$ ($4k - 7$ zeros). If $i = 2^p(2q + 1) - 1$, $p, q > 0$ and $2^pq = s + 3 \geqslant 3$, let x_i be the class of $RP(2^p, 3, 1, \ldots, 1, 0)$ (s ones). The criterion of [2, Lemma 3.4] shows that these are indecomposable in $\mathfrak{N}_*$, giving generators except in dimensions 2, 4 and 5. Using quaternionic multiplication provides a locally free S^3 action on any $RP(3, n_2, \ldots, n_k)$.

By examining the characteristic number arguments in [2, Proposition 7.2] it suffices to show that the classes x_4 and x_5 may be chosen so that x_4^2, x_4x_5 and x_5^2 are represented by manifolds with locally free S^3 action, where $w_4(x_4) = 0$.

To show that appropriate classes exist let:

(a) $M^8 = RP(3, 3, 0) \cup RP(\lambda \oplus 4)$ where λ is the canonical line bundle over $RP(3, 0)$,

(b) $M^9 = RP(3, 2, 0, 0, 0) \cup RP(3, 2, 1, 0)$, and

(c) $M^{10} = RP(3, 1, 1, 0, 0, 0) \cup RP(3, 5, 0)$.

These manifolds are decomposable and have the characteristic numbers $s_{4,4}$, $s_{4,5}$ and $s_{5,5}$ nonzero. The relation between s-numbers and decomposability in $\mathfrak{N}_*$ completes the argument.

Note. If M^n fibers over $RP(3)$, the tangent bundle of M has 3 sections. These examples show that the set of classes in $\mathfrak{N}_*$ which fiber over $RP(3)$ is the same ideal.

Turning to actions of $(S^1)^3$ and making use of the analysis from [2, Proposition 7.2] as we did in Proposition 3.1, we see that the classes in $\mathfrak{N}_*$ represented by manifolds with a locally free $(S^1)^3$ action coincides with the set of classes described for S^3 actions if and only if there is an indecomposable M^6 with a locally free $(S^1)^3$ action.

In fact, there is no such manifold, or more precisely, we have

PROPOSITION 3.3. *If the closed manifold M^6 admits a locally free $(S^1)^3$ action, then M^6 bounds.*

Note. $RP(3, 0, 0, 0)$ is indecomposable and has three linearly independent vector fields as noted above.

PROOF. Since $(S^1)^3$ is connected, it suffices to show that each component of M bounds, and so we may assume M connected. Without loss of generality, one may assume that the fixed set F of $Z_2 \times \{1\} \times \{1\} = Z_2$ on M is a proper subset. (If F is empty, M bounds, and if $F = M$, $(S^1)^3/Z_2 \cong (S^1)^3$ acts on M. Since $Z_{2^s} \times \{1\} \times \{1\}$ cannot act trivially for all s in a locally free action, at some stage the fixed set cannot be all of M.)

Then F consists of a disjoint union of submanifolds on each of which $(S^1)^3$ will act locally freely. Thus, $F = F^5 \cup F^4 \cup F^3$ and M is bordant to $RP(\nu \oplus \underline{1})$ where ν is the normal bundle of F in M. This manifold $RP(\nu \oplus \underline{1})$ is the union of the portions over the F^i, so it will suffice to see that each bounds.

In fact, more is true. $\mathfrak{N}_6 \cong (Z_2)^3$ and is detected by the numbers associated to w_6, w_4w_2 and w_3^2. Since M^6 has w_6 and w_4w_2 zero, it will suffice to show that w_3^2 is zero on each portion of $RP(\nu \oplus \underline{1})$.

For F^5, $RP(\nu \oplus \underline{1})$ is a circle bundle over F^5, so bounds.

Each component of F^3 is the form $(S^1)^3/G_x$, where G_x is an isotropy group, so is a copy of $(S^1)^3$. Looking at $RP(\nu \oplus \underline{1}) = N$ where ν is a 3-plane bundle over $(S^1)^3$,

$$w(N) = (1 + c)^4 + v_1(1 + c)^3 + v_2(1 + c)^2 + v_3(1 + c)$$

where $c = w_1(\lambda)$, $v_i = w_i(\nu)$ and $c^4 + v_1c^3 + v_2c^2 + v_3c = 0$. Then $w_3(N) = v_1c^2 + v_3$ so $w_3(N)^2 = 0$ since $v_1^2 = 0$ and $v_3^2 = 0$ in $(S^1)^3$.

Now looking at a component of F^4, call it P, $(S^1)^3$ acts locally freely on P, so $\tau(P) = l \oplus \underline{3}$ where τ is the tangent bundle and l is a line bundle. Letting $N = RP(\nu \oplus \underline{1})$ where ν is the normal bundle of P in M, one has

$$w(N) = (1 + \alpha)\left[(1 + c)^3 + v_1(1 + c)^2 + v_2(1 + c)\right]$$

where $\alpha = w_1(P)$, $c = w_1(\lambda)$ and $v_i = w_i(\nu)$, giving

$$w_3(N) = c^2\alpha + v_2\alpha$$

since $c^3 + v_1c^2 + v_2c = 0$. Then

$$\begin{aligned} w_3^2[N] &= \left(c^4\alpha^2 + w_2^2\alpha^2\right)[N] \\ &= c^4\alpha^2[N] = c^2\left(v_2 + v_1^2\right)\alpha^2[N] \\ &= \left(v_2 + v_1^2\right)\alpha^2[P] = \left(v_2 + v_1^2\right)^2[P], \end{aligned}$$

the latter equality following from the fact that α^2 is the 2-dimensional Wu class in P.

From the results of Mostert [1] is is immediately clear that $P = Q^3 \times S^1$, for some manifold Q. (To see this, note that by Mostert's analysis P, which has a locally free $(S^1)^3$ action can be constructed quite explicitly. The principal orbits are copies of $(S^1)^3/G_x \cong (S^1)^3 = H$ and either $P = H \times (P/H) = H \times (S^1)$ or P is constructed using one or two copies of Z_2 inside H. Using the fact that H has a unique copy of $(Z_2)^3$ inside it, the Z_2's can be taken as standard factors in H.) But in $Q^3 \times S^1$ the squaring operation on $H^2(Q^3 \times S^1; Z_2)$ to $H^4(Q^3 \times S^1; Z_2)$ is trivial, and so

$$w_3^2[N] = \left(v_2 + v_1^2\right)^2[P] = 0.$$

Thus, M^6 bounds.

Note. To our knowledge the analogous question of fibering an indecomposable M^6 over $(S^1)^3$, as raised in [2], is still unsettled.

References

1. P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. **65** (1957), 447–455.

2. R. E. Stong, *On fibering of cobordism classes*, Trans. Amer. Math. Soc. **178** (1973), 431–447.

3. ______, *Subbundles of the tangent bundle*, Trans. Amer. Math. Soc. **200** (1974), 185–197.

4. H. E. Winkelnkemper, *Locally free actions and Stiefel-Whitney numbers*, Proc. Sympos. Pure Math., vol. 27, part I, Amer. Math Soc., Providence, R.I., pp. 323–330.

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