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ISOMORPHISMS OF SUMS OF BOOLEAN ALGEBRAS

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ABSTRACT. We prove that any Boolean algebra is a homomorphic image of a Boolean algebra B isomorphic to $B + B + B$ but not to $B + B$.

In [5], W. Hanf constructed a Boolean algebra H such that H is isomorphic to the direct product $H \times H \times H$ but not to $H \times H$. Hanf's result was generalized in [2] as follows. For any Abelian group $(G, +)$ there exists a collection of nonisomorphic Boolean algebras $\{B(g) | g \in G\}$ such that $B(g) \times B(g')$ is isomorphic to $B(g + g')$ for all $g, g' \in G$.

In [4], P. R. Halmos examines various problems concerning isomorphisms of products of Boolean algebras and asks the corresponding questions for sums in place of products. In [3] and [6], two nonisomorphic Boolean algebras, say A and B, are constructed such that the sum $A + A$ is isomorphic to $B + B$ (in [6], A and B are countable). Here, we construct an analogy (for sums in place of products) of Hanf's result. Our purpose is to prove the following more general

THEOREM. *For any finite Abelian group $(G, +)$ and for any Boolean algebra B there exists a collection of Boolean algebras $\{B(g) | g \in G\}$ such that*:

(1) *if $g \neq g'$, then $B(g)$ is not isomorphic to $B(g')$*;

(2) *$B(g + g')$ is isomorphic to the sum $B(g) + B(g')$ for all $g, g' \in G$*;

(3) *B is a homomorphic image of any $B(g)$, $g \in G$.*

We prove the dual form of the Theorem, namely that, for any Boolean space Y, there exists a collection $\{Y(g) | g \in G\}$ of nonhomeomorphic Boolean spaces such that Y can be embedded in any $Y(g)$ and, for all $g, g' \in G$, $Y(g + g')$ is homeomorphic to $Y(g) \times Y(g')$.

REMARK. In [7] and [8], the following assertion is proved: For any commutative semigroup $(S, +)$ there exists a collection of nonhomeomorphic topological spaces $\{X(s) | s \in S\}$ such that, for all $s, s' \in S$, $X(s + s')$ is homeomorphic to $X(s) \times X(s')$. By [1], the spaces $X(s)$ can be chosen to be topological sums (i.e. disjoint unions as closed-and-open subsets) of Boolean spaces. For which semigroups $(S, +)$, can the spaces $X(s)$ be chosen to be Boolean? The full answer is unknown. (By the theorem presented here, any finite Abelian group has this property.)

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Proof of the Theorem. 1. Denote by N the set of all nonnegative integers. Given a Boolean space Y, choose a cardinal α such that Y is a subspace of the generalized Cantor discontinuum 2^α (if Y is a one-point space, we can choose $\alpha = 0$). Choose a sequence of uncountable regular cardinals $\{\beta_k\}$ such that $2^\alpha < \beta_0 < \beta_1 \ldots$. Given $k \in N$, denote by V_k a one-point compactification of the topological sum of β_k copies of the space 2^α, the added point is denoted by τ_k. If $n \in N \setminus \{0\}$, denote by V_k^n the space $V_k \times \cdots \times V_k$ n-times, V_k^0 is a one-point space. For any sequence $l = \{l_k\} \in N^N$, let $V(l) = \prod_{k=0}^{\infty} V_k^{l_k}$. Clearly $V(l)$ is a Boolean space.

2. Let P be a topological space, β a cardinal number. Denote by $\chi(x)$ the minimal cardinality of a local base of P at x. Following [1], we define a topological invariant $\lambda_\beta(x)$ inductively as follows. $\lambda_\beta(x) = 0$ iff $\chi(x) < \beta$; for $n \geqslant 1$, $\lambda_\beta(x) \leqslant n$ iff $\chi(x) = \beta$ and any discrete set D, such that card $D = \beta$ and $D \cup \{x\}$ is compact, contains a subset D' with card $D' = \beta$ and $\lambda_\beta(y) \leqslant n - 1$ for all $y \in D'$; $\lambda_\beta(x) = n$ iff $\lambda_\beta(x) \leqslant n$ but not $\lambda_\beta(x) \leqslant n - 1$; $\lambda_\beta(x) = \infty$ otherwise.

3. Lemma. *Let l be in N^N, x in $V(l)$. Let $i \in N$ be given. The point x has precisely t coordinates equal to τ_i and no coordinate equal to some τ_k with $k > i$ iff $\lambda_{\beta_i}(x) = t \in N$.*

Proof. The point x has a coordinate equal to some τ_k with $k > i$ iff $\chi(x) > \beta_i$. In this case $\lambda_{\beta_i}(x) = \infty$. Let us suppose that no coordinate of x is equal to some τ_k with $k > i$, hence $\chi(x) \leqslant \beta_i$. Denote by t the number of coordinates of x which are equal to τ_i. Clearly $t \leqslant l_i$, hence t is in N. We have to prove $\lambda_{\beta_i}(x) = t$. We have

$$t = 0 \Leftrightarrow \chi(x) < \beta_i \Leftrightarrow \lambda_{\beta_i}(x) = 0.$$

Now we proceed by induction. Denote by $K(i, s)$ the set of all points $y \in V(l)$ such that y has precisely s coordinates equal to τ_i and none equal to some τ_k with $k > i$. Let us suppose that $\lambda_{\beta_i}(y) = s \Leftrightarrow y \in K(i, s)$ whenever $s \leqslant t - 1$.

If $x \in K(i, t)$ with $t > 0$, choose a coordinate, say c, such that the cth coordinate x_c of x is equal to τ_i. Choose a discrete subset D_i in the space V_i such that card $D_i = \beta_i$ and $D_i \cup \{\tau_i\}$ is compact. For any $d \in D_i$ denote by $y(d)$ the point of $V(l)$ such that $y(d)$ has the same coordinates as x except the cth one. Here, x_c is replaced by d. Clearly, $D = \{y(d) | d \in D_i\}$ is a discrete subset of $V(l)$, $D \cup \{x\}$ is compact and $\lambda_{\beta_i}(y(d)) = t - 1$, by the induction hypothesis. Hence, $\lambda_{\beta_i}(x) \leqslant t - 1$ is not true. It remains to prove $\lambda_{\beta_i}(x) \leqslant t$. Clearly, $\chi(x) = \beta_i$. Let D be a discrete subset of $V(l)$ such that $D \cup \{x\}$ is compact and card $D = \beta_i$. Let L be the set of all $y \in V(l)$ such that ath coordinate y_a of y is equal to x_a for all those coordinates x_a of x for which $x_a \neq \tau_i$. One can verify easily that x has a system of neighbourhoods, say $\mathfrak{U}$, such that card $\mathfrak{U} < \beta_i$ and $\bigcap \mathfrak{U} \subset L$. Since $D \setminus \mathfrak{U}$ is finite for any neighbourhood $\mathfrak{U}$ of x, there exists a set $E \subset D$ such that card $E < \beta_i$ and

$D \setminus E \subset L$. Then any $y \in D' = D \setminus E$ has at most $t - 1$ coordinates equal to τ_i, hence $\lambda_{\beta_i}(y) \leqslant t - 1$.

4. LEMMA. *Let sequences $l, m \in N^N$ differ in infinitely many members. Then no nonvoid open subset of $V(l)$ is homeomorphic to an open subset of $V(m)$.*

PROOF. Let $\mathcal{O} \subset V(l)$ be open and nonempty. Then $\mathcal{O}$ contains a set $\prod_{i=0}^{\infty} W_i^{l_i}$, where all the W_i's are nonempty, open in the V_i's and there exists i_0 such that $W_i = V_i$ for all $i \geqslant i_0$. For $i \geqslant i_0$, l_i is the maximal t such that $\lambda_{\beta_i}(x) = t$ for some $x \in \mathcal{O}$, by 3. Analogously, for any nonempty open $\mathcal{O}' \subset V(m)$ there exists i_1 such that, for all $i \geqslant i_1$, m_i is the maximal t such that $\lambda_{\beta_i}(x) = t$ for some $x \in \mathcal{O}'$. Since λ_{β_i} is a topological invariant, no such $\mathcal{O} \subset V(l)$ can be homeomorphic to such $\mathcal{O}' \subset V(m)$.

5. Denote by $+$ the usual addition in N. For $\{m_i\}$, $\{l_i\}$ in N^N, put $\{m_i\} + \{l_i\} = \{m_i + l_i\}$. For $M, L \subset N^N$, put

$$M + L = \{m + l \mid m \in M, l \in L\}.$$

Let a natural number $n \geqslant 3$ be given. By [7], there exists a countable set $A \subset N^N$ such that

(i) $A = A + \cdots + A$ (n-times);

(ii) if $c, c' \in \{1, \ldots, n-1\}, c \neq c'$, then any $m \in A + \cdots + A$ (c-times) differs from any $l \in A + \cdots + A$ (c'-times) in infinitely many members.

Denote by X the topological sum of $\aleph_0$ copies of every space $V(l)$ with $l \in A$. By (i), X is homeomorphic to X^n.

6. LEMMA. *There exists a Boolean space Z homeomorphic to Z^n and containing X as an open dense subspace.*

PROOF. Let h be a homeomorphism of X onto X^n. Put $X_0 = X$, $X_1 = X^n$, $h_{1,0} = h$. Denote by Z_0 the Čech-Stone compactification βX of X and by i_0: $X \to \beta X$ the embedding. Put $Z_1 = Z_0^n$, $\iota_1 = \iota_0^n$ (this means $\iota_1(x_1, \ldots, x_n) = (\iota_0(x_1), \ldots, \iota_0(x_n))$). Denote by $f_{1,0}$: $\beta X \to (\beta X)^n$ the continuous extension of h, i.e. $f_{1,0} \circ \iota_0 = \iota_1 \circ h_{1,0}$.

Now we proceed by transfinite induction. Let c be an ordinal and let chains (= direct spectra) $(X_a, h_{b,a})$, $(Z_a, f_{b,a})$ and a collection $\{\iota_a\}$ be constructed, $a \leqslant b < c$, such that:

(a) all the $h_{b,a}$'s are homeomorphisms of X_a onto X_b (hence all the X_a's are homeomorphic to X); all the Z_a's are Boolean spaces, all the mappings $f_{b,a}$: $Z_a \to Z_b$ are continuous and surjective;

(b) for all $a < c$, ι_a: $X_a \to Z_a$ is a homeomorphism of X_a onto an open dense subset of Z_a;

(c) $\iota_b \circ h_{b,a} = f_{b,a} \circ \iota_a$ for all $a \leqslant b < c$;

(d) $h_{b,a}^n = h_{b+1,b} \circ h_{b,a+1}$, $f_{b,a}^n = f_{b+1,b} \circ f_{b,a}$ whenever $a < b$ and $b + 1 < c$.

Let us notice that (a)–(d) are fulfilled in the above case $c = 2$.

I. Let c be a limit ordinal. $(X_c, \{h_{c,a} \mid a < c\})$ is defined as a colimit of $(X_a, h_{b,a})$ in the category of all topological spaces, $(Z_c, \{f_{c,a} \mid a < c\})$ is defined as a colimit of $(Z_a, f_{b,a})$ in the category of all Boolean spaces. Clearly, all h_{ca}'s are homeomorphisms of X_a onto X_c, all $f_{c,a}$'s are surjective. Let $\iota_c\colon X_c \to Z_c$ be the unique continuous mapping such that

$$f_{c,a} \circ \iota_a = \iota_c \circ h_{c,a}$$

for all $a < c$. We have to show that ι_c is a homeomorphism of X_c onto an open dense subset of Z_c. Let $(Z'_c, \{f'_{c,a} | a < c\})$ be a colimit of $(Z_a, f_{b,a})$ in the category of all topological spaces and $\iota'_c\colon X_c \to Z'_c$ the mapping such that $f'_{c,a} \circ \iota_a = \iota'_c \circ h_{c,a}$ for all $a < c$. By (a)–(c), $(f_{b,a})^{-1}(x)$ is a one-point subset of Z_a whenever $x \in \iota_b(X_b)$. Hence ι'_c is a homeomorphism of X_c onto an open subset of Z'_c. Now, Z_c is obtained by identifications of some points in Z'_c (namely, all the pairs of nonseparated points and also all the components having more than one point; these identifications must be repeated sufficiently many times). Since $\iota'_c(X_c)$ is locally compact, Hausdorff, zero-dimensional and open in Z'_c, all these identifications are made in $Z'_c \setminus \iota'_c(X_c)$. Consequently, ι_c is one-to-one and $\iota_c(X_c)$ is open in Z_c. It is dense in Z_c because $\iota_0(X_0)$ is dense in Z_0 and $f_{c,0}$ is surjective. Thus, (a)–(c) are verified for $a \leqslant b \leqslant c$; (d) is fulfilled trivially.

II. Let c be an isolated ordinal. (a) Let us suppose $c = d + 2$ for an ordinal d. We put

$$X_c = X^n_{d+1}, \qquad Z_c = Z^n_{d+1}, \qquad \iota_c = \iota^n_{d+1},$$
$$h_{d+2,d+1} = h^n_{d+1,d}, \qquad f_{d+2,d+1} = f^n_{d+1,d}.$$

Then (a)–(d) are fulfilled for $a \leqslant b \leqslant c$, obviously.

(b) Let us suppose $c = d + 1$ for a limit ordinal d. We put $X_c = X^n_d$, $Z_c = Z^n_d$, $\iota_c = \iota^n_d$. We have to define $h_{c,d}$ and $f_{c,d}$. Denote by

$$\pi_{i,a}\colon X_{a+1} = X^n_a \to X_a, \qquad \rho_{i,a}\colon Z_{a+1} = Z^n_a \to Z_a$$

the projections, $i = 1, \dots, n$. Since $\iota_{a+1} = \iota^n_a$, we have $\iota_a \circ \pi_{i,a} = \rho_{i,a} \circ \iota_{a+1}$ for $i = 1, \dots, n$. By (d), the chain $(X^n_a, h^n_{b,a})$ (for $a \leqslant b < d$) has the same colimit X_d as $(X_a, h_{b,a})$ and, analogously, $(Z^n_a, f^n_{b,a})$ has the same colimit Z_d as $(Z_a, f_{b,a})$. Consequently, there exist mappings

$$\xi_{i,d}\colon X_d \to X_d, \quad \eta_{i,d}\colon Z_d \to Z_d, \qquad i = 1, \dots, n,$$

such that

$$\xi_{i,d} \circ h_{d,a+1} = h_{d,a} \circ \pi_{i,a}, \quad \eta_{i,d} \circ f_{d,a+1} = f_{d,a} \circ \rho_{i,a}$$

for all $a < d$ and $i = 1, \dots, n$. Now, let

$$h_{c,d}\colon X_d \to X^n_d = X_c, \quad f_{c,d}\colon Z_d \to Z^n_d = Z_d$$

be the mappings such that

$$\pi_{i,d} \circ h_{c,d} = \xi_{i,d}, \quad \rho_{i,d} \circ f_{c,d} = \eta_{i,d} \quad \text{for } i = 1, \dots, n.$$

We have to prove (c) and (d) for $a \leqslant b \leqslant c$. If $a + 1 < d$, then

$$\begin{aligned}\eta_{i,d} \circ \iota_d \circ h_{d,a+1} &= \eta_{i,d} \circ f_{d,a+1} \circ \iota_{a+1} \\ &= f_{d,a} \circ \rho_{i,a} \circ \iota_{a+1} = f_{d,a} \circ \iota_a \circ \pi_{i,a} \\ &= \iota_d \circ h_{d,a} \circ \pi_{i,a} = \iota_d \circ \xi_{i,d} \circ h_{d,a+1}.\end{aligned}$$

Since $h_{d,a+1}$ is a homeomorphism of X_{a+1} onto X_d, we obtain $\eta_{i,d} \circ \iota_d = \iota_d \circ \xi_{i,d}$ for all $i = 1, \ldots, n$. This easily implies $f_{c,d} \circ \iota_d = \iota_d^n \circ h_{c,d}$, hence (c) is fulfilled. Now,

$$\pi_{i,d} \circ h_{d,a}^n = h_{d,a} \circ \pi_{i,a}$$

for $i = 1, \ldots, n$ and

$$\pi_{i,d} \circ h_{c,d} \circ h_{d,a+1} = \xi_{i,d} \circ h_{d,a+1} = h_{d,a} \circ \pi_{i,a}$$

for $i = 1, \ldots, n$. Hence, $h_{d,a}^n = h_{c,d} \circ h_{d,a+1}$, and analogously for the f's. Thus, (d) is fulfilled. To verify (a) and (b) for $a \leqslant b \leqslant c$, it suffices to prove that $f_{c,d}$ is surjective and that $h_{c,d}$ is a homeomorphism. This straightforward proof is omitted. The induction is finished.

Since all the $f_{b,a}$'s are surjective, there exists an ordinal c such that $f_{c+1,c}$: $Z_c \to Z_c^n$ is a homeomorphism. Thus, $Z = Z_c$ has the required properties.

7. Now, we finish the proof of the Theorem. The space $V(l)$, constructed in 1, depends not only on the sequence $l \in N^N$ but also on the sequence $\mathbf{B} = \{\beta_k\}$ of cardinals chosen in 1. We shall denote it by $V(l, \mathbf{B})$. The space X, constructed in 5, and the space Z, constructed in 6, depend on $\mathbf{B}$, on the given $n > 2$ and on the set $A \subset N^N$ fulfilling (i) and (ii) from 5. We shall denote these spaces by $X(A, \mathbf{B})$ and $Z(A, \mathbf{B})$.

Now, let a finite Abelian group G be given. Express G as a product $\prod_{j=1}^s C_{n_j}$, where C_{n_j} is a finite cyclic group of the order $n_j \geqslant 2$. Express C_{n_j} as $\{1, \ldots, n_j\}$ with the usual addition and the congruence modulo n_j in place of identity. Choose a sequence $\mathbf{B} = \{\beta_k\}_{k=0}^{\infty}$ of cardinals as in 1 and put $\mathbf{B}_j = \{\beta_{s\cdot k+j-1}\}_{k=0}^{\infty}$. Denote by A_j a subset of N^N fulfilling (i) and (ii) from 5 with $n = n_j + 1$. If $g = (c_1, \ldots, c_s)$ is an element of G (i.e. $c_j \in \{1, \ldots, n_j\}$), put

$$X(g) = \prod_{j=1}^{s} \left(X(A_j, \mathbf{B}_j)\right)^{c_j},$$

$$Y(g) = \prod_{j=1}^{s} \left(Z(A_j, \mathbf{B}_j)\right)^{c_j}.$$

Clearly, $Y(g + g')$ is homeomorphic to $Y(g) \times Y(g')$ for all $g, g' \in G$. Now we have to prove that $Y(g)$ is not homeomorphic to $Y(g')$ for $g \neq g'$. Since any $X(A_j, \mathbf{B}_j)$ is an open dense subset of $Z(A_j, \mathbf{B}_j)$ (see 6), $X(g)$ is an open dense subset of $Y(g)$. Hence, if $Y(g)$ is homeomorphic to $Y(g')$, then an open nonvoid subset of $X(g)$ must be homeomorphic to an open subset of $X(g')$. We show that this is impossible for $g \neq g'$. If $l_j \in N^N, j = 1, \ldots, s$, denote by $l_1 * \cdots * l_s$ the sequence $m \in N^N$ such that the $(s \cdot k + j - 1)$th member of m is the kth member of l_j. Since $X(A_j, \mathbf{B}_j)$ is a topological sum of the spaces $V(l, \mathbf{B}_j)$ with $l \in A_j$, each in $\aleph_0$ copies (see 5), $X(g)$ is a topological sum of $\prod_{j=1}^s V(l_j, \mathbf{B}_j)$ with $l_j \in A_j + \cdots + A_j$ (c_j-times) for all $j = 1, \ldots, s$ (each in $\aleph_0$ copies again). Clearly, $\prod_{j=1}^s V(l_j, \mathbf{B}_j)$ is homeomorphic to $V(l_1 * \cdots * l_s, \mathbf{B})$. If $g = (c_1, \ldots, c_s) \neq g' = (c_1', \ldots, c_s')$, then $c_i \neq c_i'$ for some $i = 1, \ldots, s$. Then any $l_i \in A_i + \cdots + A_i$ (c_i-times) differs from any

$l_i' \in A_i + \cdots + A_i$ (c_i'-times) in infinitely many members, by 5. Hence any $m = l_1 * \cdots * l_s$ with $l_j \in A_j + \cdots + A_j$ (c_j-times) differs from any $m' = l_1' * \cdots * l_s'$ with $l_j' \in A_j + \cdots + A_j$ (c_j'-times) in infinitely many members. By 4, no nonvoid open subset of $V(m, \mathbf{B})$ is homeomorphic to an open subset of $V(m', \mathbf{B})$. Consequently, no nonvoid open subset of $X(g)$ is homeomorphic to an open subset of $X(g')$.

Concluding remarks. We can obtain results for other classes of algebras from the Theorem by means of appropriate functors. Let $\mathfrak{B}$ be the category of all Boolean algebras. In [3], full embeddings Φ_1: $\mathfrak{B} \to \mathcal{K}_1$, Φ_2: $\mathfrak{B} \to \mathcal{K}_2$, Φ_3: $\mathfrak{B} \to \mathcal{K}_3$ are constructed, where $\mathcal{K}_1$ is the variety generated by a primal algebra A, $\mathcal{K}_2$ is the variety of (0, 1)-distributive lattices, $\mathcal{K}_3$ is the category of all rings containing a fixed field A and all their A-homomorphisms. By [3], Φ_i preserves sums (in the last case, the sum is the tensor product over A). One can easily see that all the Φ_i's preserve epimorphisms. Hence, for any object B of $\mathcal{K}_i$ and any finite Abelian group $(G, +)$ there exists a collection $\{B(g) | g \in G\}$ of nonisomorphic objects of $\mathcal{K}_i$ such that $B(g + g')$ is isomorphic to the sum of $B(g)$ and $B(g')$ and B is an image of any $B(g)$ in a morphism.

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