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# ON THE VON NEUMANN ALGEBRA OF AN ERGODIC GROUP ACTION[1]

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ABSTRACT. We give a criterion that an ergodic action be amenable in terms of the operator algebra associated to it by the Murray-von Neumann construction.

The notion of an amenable ergodic action of a locally compact group was introduced by the author in [7]. (The reader is cautioned that this is not the same as the notion introduced by Greenleaf in [3]; see [7] for a discussion of this point.) In this note, we give a criterion for an action of a countable discrete group to be amenable in terms of the von Neumann algebra associated to it by the classical Murray-von Neumann construction.

We recall that J. T. Schwartz has introduced the following property of von Neumann algebras [6, p. 168]. If $A$ is a von Neumann algebra on a Hilbert space $H$, $A$ is said to have property P if for any $T \in B(H)$, the closed (weak operator topology) convex hull of $\{U^*TU | U \in U(A)\}$ contains an element of $A'$, where $U(A)$ is the unitary group of $A$, and $A'$ is, as usual, the commutant of $A$. If $G$ is a countable discrete group and $R(G)$ the von Neumann algebra generated by the right regular representation of $G$, then $R(G)$ has property P is and only if $G$ is amenable [5, Proposition 4.4.21]. More generally, Schwartz shows that if $G$ is amenable and acts ergodically on a Lebesgue space $(S, \mu)$, then $R(S \times G)$, the algebra of the Murray-von Neumann construction (described below) has property P [6, p. 198]. Conversely, if $R(S \times G)$ has property P and $\mu$ is finite and $G$-invariant, then $G$ must be amenable [6, p. 200]. The point of this note is to prove a stronger converse, without the assumption of a finite invariant measure. Namely, we show that if $R(S \times G)$ has property P, then $S$ is an amenable $G$-space (definition below). The result quoted above in the case of finite invariant measure then follows from [7, Proposition 4.4], which asserts that if $S$ is an amenable $G$-space with finite invariant measure (or mean), then $G$ itself must be amenable.

We recall the Murray-von Neumann construction. Let $S$ be a standard Borel space, $G$ a countable discrete group with a right Borel action of $G$ on $S$. We suppose $\mu$ is a probability measure on $S$ quasi-invariant and ergodic

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under $G$. We let $r(s, g)$ be the Radon-Nikodym cocycle of the action, i.e. a Borel function such that $d\mu(sg) = r(s, g)d\mu(s)$. Let $U_g$: $L^2(G) \to L^2(G)$ be the right regular representation of $G$, and define a unitary representation $\tilde{U}_g$: $L^2(S \times G) \to L^2(S \times G)$ by

$$\left(\tilde{U}_g f\right)(s, h) = f(sg, hg) r(s,g)^{1/2}.$$

Here $S \times G$ has the product measure of $\mu$ with Haar measure. Let $V_g$ be the left regular representation of $G$ on $L^2(G)$ and define $\tilde{V}_g$: $L^2(S \times G) \to L^2(S \times G)$ by

$$\left(\tilde{V}_g f\right)(s, h) = f(s, g^{-1}h).$$

If $f \in L^\infty(S)$, then $f$ defines a multiplication operator $M_f$ on $L^2(S \times G)$ by $(M_f h)(s, g) = f(s)h(s, g)$ and a multiplication operator $N_f$ by $(N_f h)(s, g) = f(sg)h(s, g)$. We let $L$ be the von Neumann algebra generated by $\{\tilde{V}_g, N_f\}$ and $R$ the von Neumann algebra generated by $\{\tilde{U}_g, M_f\}$. Then $R' = L$, and there is a unitary involution $J$ on $L^2(S \times G)$ such that $JRJ = L$ [1, pp. 137–138]. Thus $R$ and $L$ are spatially isomorphic and one will have property P if and only if the other does.

The definition of an amenable ergodic action given in [7] is based upon an "invariant section property" and is motivated by the virtual subgroup viewpoint of Mackey. We review the definition. Suppose $E$ is a separable Banach space and $\gamma$: $S \times G \to \mathrm{Iso}(E)$ is a Borel cocycle, where $\mathrm{Iso}(E)$ is the group of isometric isomorphisms of $E$ with the Borel structure of the strong operator topology. Let $E_1^*$ be the unit ball in the dual of $E$ with the $\sigma(E^*, E)$ topology. Then there is an induced adjoint cocycle $\gamma^*$: $S \times G \to \mathrm{Homeo}(E_1^*)$, $\gamma^*(s, g) = (\gamma(s, g)^*)^{-1}$. A Borel function $\phi$: $S \to E_1^*$ is called an invariant section for $\gamma$ if for each $g \in G$, $\gamma^*(s, g)\phi(sg) = \phi(s)$ for almost all $s \in S$. Now suppose that for each $s$, $A_s \subset E_1^*$ is a compact convex set, that $\{(s, x) | x \in A_s\}$ is a Borel subset of $S \times E_1^*$, and that for each $g$, $\gamma^*(s, g)A_{sg} = A_s$ for almost all $s$. Then $S$ is called an amenable $G$ space if for each such cocycle $\gamma$ and each such collection $\{A_s\}$, there is an invariant section $\phi$ with $\phi(s) \in A_s$ for almost all $s$. In [7] it is shown that any ergodic action of any amenable group is amenable, but that nonamenable groups can also have amenable actions. For example the range-closure of a cocycle [4] of an amenable action into any locally compact group is amenable [7, Theorem 3.3].

THEOREM. *If $G$ is a countable discrete group acting ergodically on $(S, \mu)$ and the von Neumann algebra $R$ (or equivalently, $L$) has property* P, *then $S$ is an amenable $G$-space.* (*Here $\mu$ is a quasi-invariant probability measure.*)

PROOF. If $f \in L^2(S \times G)$, define $f_s(g) = f(sg, g)r(s, g)^{1/2}$. Then $f_s \in L^2(G)$ for almost all $s$, and a straightforward calculation shows that $f \to \int^\oplus f_s \, d\mu$ is a unitary isomorphism of $L^2(S \times G) \cong \int_s^\oplus L^2(G)$. One further readily verifies that under this isomorphism, $\tilde{U}_g$ corresponds to $\int_s^\oplus U_g$ and that if $T \in B(L^2(S \times G))$ corresponds to the decomposable operator $\int^\oplus T_s$ in

$B(\int_s^{\oplus} L^2(G))$, then $T$ commutes with $\tilde{V}_g$ if and only if for each $g$, $V_g^{-1}T_sV_g = T_{sg}$ for almost all $s$. Since the operators $\tilde{U}_g$ and $M_f$, $f \in L^\infty(S)$, correspond to decomposable operators, it follows that every element of $R$ is decomposable with respect to this direct integral decomposition of $L^2(S \times G)$.

Now suppose that $L$ has property P. Then [5, Proposition 4.4.15] there is a linear mapping $P$: $B(L^2(S \times G)) \to R$ $(= L')$ such that

(i) $\|P\| \leqslant 1$, $P(I) = I$, $T \geqslant 0$ implies $P(T) \geqslant 0$.

(ii) $P(T) \in C(T)$, where $C(T)$ is the closed convex hull of $\{UTU^* | U \in U(L)\}$.

(iii) $P(S_1TS_2) = S_1P(T)S_2$ if $S_1, S_2 \in R$.

If $f \in L^\infty(S \times G)$, we have the multiplication operator $M_f \in B(L^2(S \times G))$, and since each element of $R$ is decomposable, we can write $P(M_f) = \int^{\oplus} T_s^f \, d\mu$. For $f \in L^\infty(S \times G)$, write $(f \cdot g)(s, h) = f(sg, hg)$. Then $\tilde{U}_g P(M_f)\tilde{U}_g^{-1} = P(\tilde{U}_g M_f \tilde{U}_g^{-1}) = P(M_{f\cdot g})$. Thus $\tilde{U}_g(\int^{\oplus} T_s^f)\tilde{U}_g^{-1} = \int^{\oplus} T_s^{f\cdot g}$, i.e. $\int^{\oplus} U_g T_s^f U_g^{-1} = \int^{\oplus} T_s^{f\cdot g}$. It follows that for each $g$, $U_g T_s^f U_g^{-1} = T_s^{f\cdot g}$ for almost all $s$. We also note that since $\int^{\oplus} T_s^f \in L'$, it follows from the remarks in the preceding paragraph that for each $g$, $V_g^{-1}T_s^f V_g = T_{sg}^f$ a.e.

For $f \in L^\infty(S \times G)$, define $\sigma(f)(s) = \langle T_s^f \chi_e | \chi_e \rangle$ where $\chi_e \in L^2(G)$ is the characteristic function of the identity. Then $\sigma$: $L^\infty(S \times G) \to L^\infty(S)$ and this map has the following properties:

(i) $\|\sigma(f)\|_\infty \leqslant \|f\|_\infty$.

(ii) $\sigma(1) = 1$.

(iii) If $f \geqslant 0$, $\sigma(f) \geqslant 0$.

(iv) If $A \subset S$ is measurable, $\sigma(f\chi_{A \times G}) = \sigma(f)\chi_A$.

(v) $\sigma(f \cdot g) = \sigma(f) \cdot g$.

Properties (i)–(iv) follow in a straightforward manner from the properties of the map $P$ listed above and some elementary properties of direct integrals of operators. To see (v), note that for each $g$ and almost all $s$,

$$\begin{aligned}\sigma(f \cdot g)(s) &= \left\langle T_s^{f\cdot g}\chi_e | \chi_e \right\rangle = \left\langle U_g T_s^f U_g^{-1}\chi_e | \chi_e \right\rangle \\ &= \left\langle U_g V_g T_{sg}^f V_g^{-1} U_g^{-1}\chi_e | \chi_e \right\rangle = \left\langle T_{sg}^f V_g^{-1} U_g^{-1}\chi_e | V_g^{-1} U_g^{-1}\chi_e \right\rangle \\ &= \left\langle T_{sg}^f \chi_e | \chi_e \right\rangle = \sigma(f)(sg).\end{aligned}$$

We now demonstrate how the map $\sigma$ can be used to show that $S$ is amenable. Suppose $E$, $\gamma$, and $\{A_s\}$ are as in the discussion preceding the statement of the theorem. Since $\{(s, A_s)\}$ is Borel, there is a measurable function $b$: $S \to E_1^*$ such that $b(s) \in A_s$ for almost all $s \in S$. Define $F$: $S \times G \to E_1^*$ by $F(s, g) = \gamma^*(s, g^{-1})b(sg^{-1})$. Then for each $\theta \in E$, $(s, g) \to \langle \theta, F(s, g) \rangle$ is in $L^\infty(S \times G)$ (where $\langle \, , \, \rangle$ is the duality of $E$ and $E^*$), and the map $E \to L^\infty(S)$, $\theta \to \sigma(\langle \theta, F(s, g) \rangle)$ is linear with norm $\leqslant 1$. It follows from [2, p. 582] that there is a measurable function $a$: $S \to E_1^*$ such that $\sigma(\langle \theta, F(s, g) \rangle)(s) = \langle \theta, a(s) \rangle$ a.e. We claim that $a(s)$ is the required invariant section.

LEMMA. *For all essentially bounded and measurable* $\theta$: $S \to E$,

$$\sigma(\langle\theta(s), F(s,g)\rangle)(s) = \langle\theta(s), a(s)\rangle \quad a.e.$$

PROOF. Suppose first that $\theta$ is a simple function, i.e. $\theta(s) = \Sigma_1^\infty \theta_i \chi_{A_i}(s)$ where $\{A_i\}$ is a countable partition of $S$ and $\theta_i \in E$. Fix $j$. Then for almost all $s \in A_j$, by property (iv) of $\sigma$,

$$\sigma\big(\langle\theta(s), F(s,g)\rangle\big)(s) = \sigma\big(\langle\theta(s), F(s,g)\rangle\chi_{A_j\times G}\big) = \sigma\big(\langle\theta_j, F(s,g)\rangle\chi_{A_j\times G}\big)$$
$$= \sigma\big(\langle\theta_j, F(s,g)\rangle\big)(s) = \big\langle\theta_j, a(s)\big\rangle = \big\langle\theta(s), a(s)\big\rangle.$$

Since $j$ is arbitrary, this lemma holds for simple $\theta$. If $\theta$ is arbitrary, then there are simple functions $\theta_n$ with $\|\theta_n - \theta\|_\infty \to 0$ by virtue of the separability of $E$. Since $F(s,g) \in E_1^*$, $\langle\theta_n(s), F(s,g)\rangle \to \langle\theta(s), F(s,g)\rangle$ in $\| \ \|_\infty$ on $S \times G$, and by the norm continuity of $\sigma$,

$$\sigma\big(\langle\theta_n(s), F(s,g)\rangle\big) \to \sigma\big(\langle\theta(s), F(s,g)\rangle\big)$$

in $L^\infty(S)$. Clearly $\langle\theta_n(s), a(s)\rangle \to \langle\theta(s), a(s)\rangle$ a.e., and the lemma follows.

COROLLARY. *Suppose $\alpha: S \to \mathrm{Iso}(E)$ is measurable. Then for all $\theta \in E$,*

$$\sigma\big(\langle\theta, \alpha(s)^* F(s,g)\rangle\big) = \big\langle\theta, \alpha(s)^* a(s)\big\rangle.$$

PROOF. This is equivalent to $\sigma(\langle\alpha(s)\theta, F(s,g)\rangle) = \langle\alpha(s)\theta, a(s)\rangle$ which holds by the lemma.

We now show that $a(s)$ is an invariant section. Suppose $h \in G$. Then by property (v) of $\sigma$,

$$\sigma\big(\langle\theta, F(s,g)\rangle \cdot h\big)(s) = \big\langle\theta, a(s)\big\rangle \cdot h = \big\langle\theta, a(sh)\big\rangle.$$

But the first term of this equation

$$= \sigma\Big(\Big\langle\theta, \gamma^*(sh, h^{-1}g^{-1})b(sg^{-1})\Big\rangle\Big)(s)$$
$$= \sigma\Big(\Big\langle\theta, \gamma^*(s,h)^{-1}\gamma^*(s, g^{-1})b(sg^{-1})\Big\rangle\Big)(s)$$

and by the corollary, since $\gamma^*(s,h)$ is the adjoint of an isometric isomorphism, this $= \langle\theta, \gamma^*(s,h)^{-1}a(s)\rangle$. Since $E$ is separable, it follows [2, Theorem 8.17.2(c)] that $\gamma^*(s,h)^{-1}a(s) = a(sh)$ for almost all $s$, i.e. $a(s)$ is an invariant section.

Thus, to complete the proof it suffices to show $a(s) \in A_s$ for almost all $s$. Let $\{\theta_i\}$ be a countable dense subset of $E$, considered as linear functionals on $E^*$. Then the hyperplanes in $E^*$ defined by $\theta_i = q$, $q$ rational, separate all compact convex subsets of $E_1^*$ from points in $E_1^*$. Therefore, it suffices to show that for all $\theta$ and $q$, $\theta(A_s) \geqslant q$ implies $\theta(a(s)) \geqslant q$ for almost all $s$. Given $\theta$ and $q$, let $S_0 = \{s \in S \mid \theta(A_s) \geqslant q\}$. Then $S_0$ is measurable by [7, Lemma 1.7]. Suppose $\mu(S_0) > 0$. Then by property (iii) of $\sigma$,

$$\sigma\big(\langle\theta, F(s,g)\rangle\chi_{S_0\times G}\big) \geqslant \sigma(q \cdot \chi_{S_0\times G}) = q\chi_{S_0}.$$

Thus $\langle\theta, a(s)\rangle \cdot \chi_{S_0} \geqslant q\chi_{S_0}$, so $\theta(a(s)) \geqslant q$ for almost all $s \in S_0$. Since $\theta$ and $q$ are arbitrary, the theorem follows.

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