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C^*-ALGEBRAS ISOMORPHIC AFTER TENSORING[1]

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ABSTRACT. It is always true that whenever $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic C^*-algebras then $\mathfrak{M}_2 \otimes \mathfrak{A}$ and $\mathfrak{M}_2 \otimes \mathfrak{B}$ are also isomorphic, and the converse holds for many standard examples. In this note we present two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{M}_2 \otimes \mathfrak{A}$ and $\mathfrak{M}_2 \otimes \mathfrak{B}$ are isomorphic whereas $\mathfrak{A}$ and $\mathfrak{B}$ are not.

We remark that whenever $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic C^*-algebras then $\mathfrak{M}_n \otimes \mathfrak{A}$ and $\mathfrak{M}_n \otimes \mathfrak{B}$ are also isomorphic, $n = 1, 2, 3, \ldots$; the converse is true for abelian C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ because the center of $\mathfrak{M}_n \otimes \mathfrak{A}$ (respectively $\mathfrak{M}_n \otimes \mathfrak{B}$) is isomorphic to $\mathfrak{A}$ (respectively $\mathfrak{B}$). It follows also from the classification theory of [3] that if $\mathfrak{A}$ and $\mathfrak{B}$ are uniformly hyperfinite algebras such that $\mathfrak{M}_2 \otimes \mathfrak{A}$ and $\mathfrak{M}_2 \otimes \mathfrak{B}$ are isomorphic, then so are $\mathfrak{A}$ and $\mathfrak{B}$. Finally, if $\mathfrak{A}$ and $\mathfrak{B}$ are perturbed block diagonal algebras then one verifies easily by applying [4, Theorem 1] that $\mathfrak{M}_n \otimes \mathfrak{A}$ isomorphic to $\mathfrak{M}_n \otimes \mathfrak{B}$ implies that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. The purpose of this note is to exhibit two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{M}_2 \otimes \mathfrak{A}$ is isomorphic to $\mathfrak{M}_2 \otimes \mathfrak{B}$ but $\mathfrak{A}$ is not isomorphic to $\mathfrak{B}$.

In what follows, $\mathcal{L}(\mathcal{H})$ (respectively $\mathcal{C}(\mathcal{H})$) shall denote the algebra of bounded (respectively compact) operators on a separable Hilbert space $\mathcal{H}$. By the essential commutant of a set $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$, denoted E.C. $(\mathcal{S})$, we shall mean the set of $T \in \mathcal{L}(\mathcal{H})$ such that $TS - ST \in \mathcal{C}(\mathcal{H})$ for every $S \in \mathcal{S}$. A 2×2 system of matrix units for a C^*-algebra $\mathcal{E}$ with identity I is defined to be a set $\{e_{ij}\}_{1 < i,j < 2}$ of elements of $\mathcal{E}$ such that $e_{ij}e_{km} = \delta_{jk}e_{im}$, $e_{ij}^* = e_{ji}$, and $e_{11} + e_{22} = I$. Finally, the natural projection of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ will be written π. Let

$$\mathfrak{A} = \{T \oplus T\colon T \in \mathcal{L}(\mathcal{H})\} + \mathcal{C}(\mathcal{H} \oplus \mathcal{H}),$$

$$\mathfrak{B} = \{0 \oplus T \oplus T\colon T \in \mathcal{L}(\mathcal{H}), 0 \in \mathcal{L}(\mathcal{K})\} + \mathcal{C}(\mathcal{K} \oplus \mathcal{H} \oplus \mathcal{H}),$$

where $\mathcal{K}$, the first coordinate space, is one dimensional.

To show that $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, we rely on the fact that two C^*-algebras of $\mathcal{L}(\mathcal{H})$ which contain $\mathcal{C}(\mathcal{H})$ are isomorphic if and only if they

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are unitarily equivalent, denoted by $\simeq$ [**1**, Theorem 1.3.4; Corollary 3, p. 20]. We note that because a unitary equivalence of two C^*-algebras induces a unitary equivalence of their essential commutants, to prove $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, it suffices to prove that E.C. $(\mathfrak{A})$ and E.C. $(\mathfrak{B})$ are not.

It is easy to verify that:

$$\text{E.C. }(\mathfrak{A}) = \left\{ \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} : \lambda_{ij} \in \mathbf{C} \right\} + \mathcal{C}(\mathcal{H} \oplus \mathcal{H}),$$

$$\text{E.C. }(\mathfrak{B}) = \left\{ 0 \oplus \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} : \lambda_{ij} \in \mathbf{C} \right\} + \mathcal{C}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}),$$

where $\mathcal{H}$ and $\mathcal{H}$ are as before and $\mathbf{C}$ denotes the set of complex numbers.

In [**2**] it is shown that any 2×2 system of matrix units in $\pi(\text{E.C.}(\mathfrak{A}))$ is the image of a 2×2 system of matrix units in E.C. $(\mathfrak{A})$ whereas no 2×2 system of matrix units in $\pi(\text{E.C. }(\mathfrak{B}))$ is, thus yielding the desired conclusion.

It remains only to show that $\mathfrak{M}_2 \otimes \mathfrak{A}$ and $\mathfrak{M}_2 \otimes \mathfrak{B}$ are isomorphic. In fact, we remark that slight modifications of our argument will show that $\mathfrak{M}_n \otimes \mathfrak{A}$ and $\mathfrak{M}_n \otimes \mathfrak{B}$ are isomorphic if and only if n is even–although we do not need this here. An observation which we isolate to summon repeatedly is that $\mathfrak{M}_n \otimes \mathcal{L}(\mathcal{H})$ (respectively $\mathfrak{M}_n \otimes \mathcal{C}(\mathcal{H})$) is isomorphic to $\mathcal{L}(\mathcal{H})$ (respectively $\mathcal{C}(\mathcal{H})$) whenever $\mathcal{H}$ is an infinite dimensional Hilbert space.

So, we concretely represent $\mathfrak{M}_2 \otimes \mathfrak{B}$ as

$$(1) \qquad \left\{ \begin{bmatrix} 0 & & & 0 & & \\ & T_{11} & & & T_{12} & \\ & & T_{11} & & & T_{12} \\ 0 & & & 0 & & \\ & T_{21} & & & T_{22} & \\ & & T_{21} & & & T_{22} \end{bmatrix} : \begin{matrix} T_{ij} \in \mathcal{L}(\mathcal{H}) \\ 0 \in \mathcal{L}(\mathcal{H}) \end{matrix} \right\}$$

$$+ \mathcal{C}((\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}))$$

which is unitarily equivalent to

$$(2) \qquad \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \oplus \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : T_{ij} \in \mathcal{L}(\mathcal{H}) \right\}$$

$$+ \mathcal{C}((\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H}))$$

by permuting the coordinate spaces. By employing the observation above, we realize (2) as unitarily equivalent to

$$(3) \qquad \{(0 \oplus T) \oplus (0 \oplus T) : 0 \in \mathcal{L}(\mathcal{H}), T \in \mathcal{L}(\mathcal{H}')\}$$
$$+ \mathcal{C}(\mathcal{H} \oplus \mathcal{H}' \oplus \mathcal{H} \oplus \mathcal{H}'),$$

where $\mathcal{H}'$ is identified with $\mathcal{H} \oplus \mathcal{H}$.

Because $\dim(\mathcal{H}) < \infty$ and our algebra contains all of the compact operators on the underlying space, (3) is equal to

$$\{S \oplus S \colon S \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')\} + \mathcal{C}((\mathcal{H} \oplus \mathcal{H}') \oplus (\mathcal{H} \oplus \mathcal{H}')), \tag{4}$$

which by the observation mentioned above is unitarily equivalent to

$$\left\{\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \oplus \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \colon T_{ij} \in \mathcal{L}(\mathcal{H})\right\} \tag{5}$$
$$+ \mathcal{C}((\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H})).$$

By interchanging the second and third coordinate spaces, we recognize (5) as

$$\left\{\begin{pmatrix} T_{11} & 0 & T_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ T_{21} & 0 & T_{22} & 0 \\ 0 & T_{21} & 0 & T_{22} \end{pmatrix} \colon T_{ij} \in \mathcal{L}(\mathcal{H})\right\} + \mathcal{C}((\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H}))$$

which is the concrete representation of $\mathfrak{M}_2 \otimes \mathfrak{A}$. $\square$

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