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# A MEASURE THEORETIC VARIANT OF BLUMBERG'S THEOREM

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ABSTRACT. It is the purpose of this note to present a measure theoretic variant of Blumberg's theorem about continuous restrictions of arbitrary real functions.

Henry Blumberg [1] proved a theorem in 1922, which in its most basic form is stated as follows:

THEOREM A. *For every $f: I \to R$, there exists $D \subset I$, $D$ dense in $I$, such that $f|D$ is continuous.*

$I$ is the interval [0, 1] and $R$ denotes the reals. Blumberg observed [2] that the set $D$ cannot be made to have cardinality $c$ because of the function $f: I \to R$ of Sierpiński and Zygmund [6] which has no continuous restriction of cardinality $c$. However, the author established theorems [3] from which the following results:

THEOREM B. *For every $f: I \to R$, there exists $W \subset I$, $W$ $c$-dense in $I$, such that $f|W$ is pointwise discontinuous (PWD).*

$W$ is *$c$-dense in $I$* if and only if every subinterval of I contains $c$-many points of $W$. A function $g$ with domain $W$ is PWD if and only if there exists $D \subset W$, $D$ dense in $W$, such that $g$ is continuous at each element of $D$.

P. Erdös recently asked the author if something could be done to make the set $W$ in the conclusion of Theorem B large relative to Lebesgue measure $\lambda$.

First, notice that it would be easy to alter the set $W$ in such a way to make it an *$M$ set* (i.e. have $\lambda^0(W) > 0$). Let $D \subset W$ be the set on which $f|W$ is continuous, let $C$ be a Cantor subset of $I$ of positive measure, and let $W' = W \cup C$. Then $\lambda^0(W') > 0$, and $f|W'$ is still continuous at each element of $D - (C \cap D)$, which is dense in $W'$.

On the other hand, it would not be possible to make $W$ have outer measure 1 or even be $M$-dense in $I$ if $f$ is a function such as the following: let $C_1, C_2, \ldots,$ be a sequence of disjoint Cantor subsets of $I$ such that $C_1 \cup C_2 \cup \ldots$ has measure 1, and let $f(x) = n$ if $x \in C_n$. *$A$ is $M$-dense in $B$* if $A \subset B$ and every open set which intersects $B$ intersects $A$ in an $M$ set.

Next we might ask if $W$ can be made to be $M$-dense in itself and drop the

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requirement that it be dense in $I$. Dropping the requirement that $W$ be dense in $I$ will, in fact, make it possible to obtain differentiability on a dense subset of $W$ if infinite derivatives are allowed (i.e. $f$ is differentiable at $x$ if $f$ is continuous at $x$, $x$ is a limit point of $D_f$, and there is $t$, $-\infty \leqslant t \leqslant \infty$, such that if $\{x_n\}$ is a sequence of elements of $D_f - \{x\}$ converging to $x$, then $\{(f(x) - f(x_n))/(x - x_n)\}$ converges to $t$). In [5] Ceder proved the following:

THEOREM C. *Suppose $X \subset I$ is uncountable. Then for every $f$: $X \to R$ there exists $D \subset X$, $D$ bilaterally dense in itself, such that $f|D$ is differentiable.*

In [4] the author proved the following variant of Ceder's theorem:

THEOREM D. *Suppose $X \subset I$ is not an $L_1$ set. Then for every $f$: $X \to R$ there exist $W \subset X$ and $D \subset W$, $W$ bilaterally $c$-dense in itself and $D$ dense in $W$ such that $f|W$ is differentiable at each element of $D$.*

An $L_1$ set is a countable union $M_1 \cup M_2 \cup \ldots$ such that for each $i$, every nowhere dense in $M_i$ subset of $M_i$ has cardinality less than $c$. A set is bilaterally dense ($c$-dense) ($M$-dense) in itself if and only if every closed interval which intersects it intersects it in an infinite set (set of cardinality $c$) ($M$ set). The converse of Theorem D was also shown to hold. An alteration of the proof of Theorem D will prove the following:

THEOREM E. *Suppose $X \subset I$ is an $M$ set. Then for every $f$: $X \to R$, there exist $W \subset X$ and $D \subset W$, $W$ bilaterally $M$-dense in itself and $D$ dense in $W$, such that $f|W$ is differentiable at each element of $D$.*

The proof of Theorem E can proceed almost identically with the proof of Theorem 1 of [4], with the notions "$L_1$" and "$L_2$" (which means "not $L_1$") of [4] replaced by "null" (which in this case means "of measure zero") and "$M$", respectively. Lemma 1 of [4] states that *if $x$ is an element of a bilaterally $L_2$-dense in itself set $A$, then there exists a bilaterally $c$-dense in itself nowhere dense in $A$ subset $N$ of $A$ containing $x$.* Indeed, the fact that the Continuum Hypothesis implies the existence of a Lusin set, so that the conclusion of Lemma 1 cannot be obtained if it is just known that $A$ is bilaterally $c$-dense in itself, is the cause of most of the difficulties encountered in proving Theorems B and D, and brought about the necessity of defining properties $L_1$ and $L_2$. The situation with respect to $M$ sets is much simpler in the sense that Lemma 1 of [4] can be replaced by the following:

LEMMA 1′. *If $x$ is an element of a bilaterally $M$-dense in itself set $A$, then there exists a bilaterally $M$-dense in itself nowhere dense in $A$ subset $N$ of $A$ containing $x$.*

PROOF. There exists a $G_\delta$ set $B$ such that $A \subset B$ and $\lambda^0(A) = \lambda(B)$. *Assume that $B \subset \mathrm{Cl}(A)$, so that $B$ is $M$-dense in itself.* It follows that if $C \subset B$ and $\lambda(C) > 0$, then $\lambda^0(A \cap C) = \lambda(C)$. For each positive integer $n$, the set $A_n = [x, x + 1/n] \cap A$ has positive outer measure. Consider a Cantor set $C_n$

of positive measure such that $C_n$ is a subset of and nowhere dense relative to $[x, x + 1/n] \cap B$. Let $C_n'$ consist of the points of density 1 of $C_n$. Then $R_n = C_n' \cap A$ will be a bilaterally $M$-dense in itself subset of $[x, x + 1/n] \cap A$ which is nowhere dense in $[x, x + 1/n] \cap A$. The set $[x - 1/n, x] \cap A$ will contain a similar set $L_n$. Then

$$N = L_1 \cup L_2 \cup \ldots \cup (x) \cup R_1 \cup R_2 \cup \ldots$$

is the desired set.

Now, Lemmas 2–6 of [4] can be altered by replacing "$L_1$" and "$L_2$" by "null" and "$M$", respectively, and the proofs will be the same. Then on stage (2) of the inductive procedure in the proof of Theorem 1 [4, p. 39], make $N_s$ be bilaterally $M$-dense in itself, and Theorem E will be proved.

## References

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