

AN ALTERNATIVE TO THE PLÜCKER RELATIONS

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ABSTRACT. It is shown how to obtain a set of homogeneous, degree m polynomials in $\binom{n}{m}$ indeterminates over a field F so that the associated algebraic variety is the set of decomposable elements in the m th Grassmann space over an n -dimensional vector space over F . The same techniques are used to produce an analogous result for the tensor product of m finite dimensional vector spaces.

Let V be an n -dimensional vector space over a field F , and denote the m th Grassmann space over V by $\wedge^m V$. (We assume throughout that $m \geq 2$.) It is well known that the set $\mathcal{D} = \{v_1 \wedge \cdots \wedge v_m : v_i \in V, i = 1, \dots, m\}$ of decomposable elements in $\wedge^m V$ may be described as the algebraic variety associated with the set of so-called Plücker relations [2], [4]. An analogous result for the tensor product $V_1 \otimes \cdots \otimes V_m$ of finite-dimensional vector spaces V_1, \dots, V_m has recently been obtained by Grone [1]. In both cases, the polynomials were homogeneous, degree 2. This paper shows how to produce alternative sets of (homogeneous, degree m) polynomials which define the same two varieties. While no claim is made that the new polynomials can be used more easily than Grone's polynomials or the notoriously cumbersome Plücker polynomials, the present approach has certain advantages: (1) The same methods work for $\wedge^m V$ and $V_1 \otimes \cdots \otimes V_m$, (2) the results have coordinate-free translations, and (3) the proofs are brief, as well as elementary.

The tensor space. Let V_i^* denote the space of linear functionals on V_i , and suppose $f_i \in V_i^*$, $i = 1, \dots, m$. The universal factorization property of "⊗" permits the definitions of unique linear transformations $c_j: V_1 \otimes \cdots \otimes V_m \rightarrow V_j$, $j = 1, \dots, m$, and $c: V_1 \otimes \cdots \otimes V_m \rightarrow F$ satisfying

$$(1) \quad c_j(v_1 \otimes \cdots \otimes v_m) = \left[\prod_{\substack{i=1 \\ i \neq j}}^m f_i(v_i) \right] v_j,$$

$$(2) \quad c(v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^m f_i(v_i),$$

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for all $v_i \in V_i, i = 1, \dots, m$. (The dependence of c_1, \dots, c_m, c on f_1, \dots, f_m will not be explicitly denoted.)

THEOREM 1. *A tensor $z \in V_1 \otimes \dots \otimes V_m$ is decomposable if and only if for every choice of $f_i \in V_i^*, i = 1, \dots, m$, we have*

$$(3) \quad c_1(z) \otimes \dots \otimes c_m(z) = [c(z)]^{m-1} z,$$

where c_1, \dots, c_m, c are defined by (1), (2).

PROOF. Obvious from the definitions and the fact that if $z \neq 0$, then there exist $f_i \in V_i^*, i = 1, \dots, m$, such that $c(z) \neq 0$. \square

By choosing bases of the spaces V_1, \dots, V_m , and by using the usual induced basis in $V_1 \otimes \dots \otimes V_m$ and the dual bases in V_1^*, \dots, V_m^* , we can convert Theorem 1 into a set of polynomial conditions on the coordinates of z which are equivalent to the decomposability of z .

Let V_i have the basis v_{i1}, \dots, v_{in_i} and let f_{i1}, \dots, f_{in_i} be the dual basis of $V_i^*, i = 1, \dots, m$. The set

$$\{v_{1\alpha(1)} \otimes \dots \otimes v_{m\alpha(m)} : 1 \leq \alpha(i) \leq n_i \text{ for } i = 1, \dots, m\}$$

is a basis of $V_1 \otimes \dots \otimes V_m$ [3] (notation: $v_{1\alpha(1)} \otimes \dots \otimes v_{m\alpha(m)} = v_\alpha^\otimes, \Gamma = \{\alpha : 1 \leq \alpha(i) \leq n_i \text{ for } i = 1, \dots, m\}$), so z has an expression

$$(4) \quad z = \sum_{\alpha \in \Gamma} a_\alpha v_\alpha^\otimes, \quad a_\alpha \in F.$$

Now, if $f_i \in \{f_{i1}, \dots, f_{in_i}\}, i = 1, \dots, m$, say $f_i = f_{i\beta(i)}, \beta \in \Gamma$, then $c(z)$ and $c_j(z), j = 1, \dots, m$, may be computed using (1), (2), (4), and the duality of the v_{ij} and f_{ij} bases:

$$(5) \quad c(z) = a_\beta,$$

$$(6) \quad c_j(z) = \sum_{t=1}^{n_j} a_{\beta_{jt}} v_{jt},$$

where $\beta_{jt} \in \Gamma$ is defined by

$$(7) \quad \beta_{jt}(i) = \begin{cases} \beta(i) & \text{if } i \neq j, \\ t & \text{if } i = j, \end{cases}$$

i.e., β_{jt} is the sequence obtained from β by replacing $\beta(j)$ by t . Compute each side of (3) using (5) and (6); (3) holds if and only if the coefficients of v_α^\otimes are equal for all $\alpha \in \Gamma$, i.e., if and only if

$$(8) \quad \prod_{i=1}^m a_{\beta_{\alpha(i)}} - a_\beta^{m-1} a_\alpha = 0.$$

Thus, (8) holds for all $\alpha, \beta \in \Gamma$ if and only if (3) holds for every selection of $f_i \in \{f_{i1}, \dots, f_{in_i}\}, i = 1, \dots, m$, and by Theorem 1, this is equivalent to the decomposability of z . We have proved

THEOREM 2. *The tensor $z = \sum_{\alpha \in \Gamma} a_\alpha v_\alpha^\otimes \in V_1 \otimes \dots \otimes V_m$ is decomposable if and only if (8) holds for all $\alpha, \beta \in \Gamma$.*

(Compare with the main result in [1].)

The Grassmann space. Let $f_i \in V^*$, $i = 1, \dots, m$. The universal factorization property of “ \wedge ” permits the definitions of unique linear transformations $c_j: \wedge^m V \rightarrow V, j = 1, \dots, m$, and $c: \wedge^m V \rightarrow F$ satisfying

$$(9) \quad c_j(v_1 \wedge \dots \wedge v_m) = \det \begin{bmatrix} f_1(v_1) \cdots f_{j-1}(v_1) & v_1 & f_{j+1}(v_1) \cdots f_m(v_1) \\ \vdots & \vdots & \vdots \\ f_1(v_m) \cdots f_{j-1}(v_m) & v_m & f_{j+1}(v_m) \cdots f_m(v_m) \end{bmatrix},$$

$$(10) \quad c(v_1 \wedge \dots \wedge v_m) = \det \begin{bmatrix} f_1(v_1) & \cdots & f_m(v_1) \\ \vdots & \vdots & \vdots \\ f_1(v_m) & \cdots & f_m(v_m) \end{bmatrix},$$

where the right side of (9) is interpreted as a vector in V , e.g.,

$$\det \begin{bmatrix} f_1(v_1) & v_1 & f_3(v_1) \\ f_1(v_2) & v_2 & f_3(v_2) \\ f_1(v_3) & v_3 & f_3(v_3) \end{bmatrix} = -\det \begin{bmatrix} f_1(v_2) & f_3(v_2) \\ f_1(v_3) & f_3(v_3) \end{bmatrix} v_1 + \det \begin{bmatrix} f_1(v_1) & f_3(v_1) \\ f_1(v_3) & f_3(v_3) \end{bmatrix} v_2 - \det \begin{bmatrix} f_1(v_1) & f_3(v_1) \\ f_1(v_2) & f_3(v_2) \end{bmatrix} v_3.$$

THEOREM 3. A skew-symmetric tensor $z \in \wedge^m V$ is decomposable if and only if for every choice of $f_i \in V^*, i = 1, \dots, m$, we have

$$(11) \quad c_1(z) \wedge \dots \wedge c_m(z) = [c(z)]^{m-1} z,$$

where c_1, \dots, c_m, c are defined by (9), (10).

PROOF. The “if” part is obvious. For the converse, suppose $z = v_1 \wedge \dots \wedge v_m$, and $f_1, \dots, f_m \in V^*$ are given. Since each $c_j(z)$ is a linear combination of v_1, \dots, v_m , $c_1(z) \wedge \dots \wedge c_m(z)$ is a (possibly zero) multiple of $v_1 \wedge \dots \wedge v_m$, say

$$(12) \quad c_1(z) \wedge \dots \wedge c_m(z) = rz, \quad r \in F.$$

From (9), it follows that

$$f_k(c_j(z)) = \begin{cases} c(z) & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

so applying c to both sides of (12) yields

$$[c(z)]^m = rc(z), \quad c(z)(r - [c(z)]^{m-1}) = 0.$$

Thus, (11) holds when $c(z) \neq 0$. If $c(z) = 0$, then the adjugate of $[f_j(v_i)]$ is singular, and it follows easily that $c_1(z), \dots, c_m(z)$ are linearly dependent, hence $c_1(z) \wedge \dots \wedge c_m(z) = 0$, so (11) still holds, and the proof is complete.

By introducing a basis of V , a dual basis of V^* , and the usual induced basis in $\wedge^m V$ [3], one may proceed in a manner analogous to that in the previous section, translating the criterion (11) into a set of m th degree polynomial conditions on the coordinates of z w.r.t. the induced basis. We choose to omit this slightly messy but straightforward calculation.

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