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AN ALTERNATIVE TO THE PLÜCKER RELATIONS

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ABSTRACT. It is shown how to obtain a set of homogeneous, degree m polynomials in $\binom{n}{m}$ indeterminates over a field F so that the associated algebraic variety is the set of decomposable elements in the mth Grassmann space over an n-dimensional vector space over F. The same techniques are used to produce an analogous result for the tensor product of m finite dimensional vector spaces.

Let V be an n-dimensional vector space over a field F, and denote the mth Grassmann space over V by $\bigwedge^m V$. (We assume throughout that $m \geqslant 2$.) It is well known that the set $\mathfrak{D} = \{v_1 \wedge \cdots \wedge v_m\colon v_i \in V,\ i = 1, \ldots, m\}$ of decomposable elements in $\bigwedge^m V$ may be described as the algebraic variety associated with the set of so-called Plücker relations [2], [4]. An analogous result for the tensor product $V_1 \otimes \cdots \otimes V_m$ of finite-dimensional vector spaces $V_1, \ldots, V_m$ has recently been obtained by Grone [1]. In both cases, the polynomials were homogeneous, degree 2. This paper shows how to produce alternative sets of (homogeneous, degree m) polynomials which define the same two varieties. While no claim is made that the new polynomials can be used more easily than Grone's polynomials or the notoriously cumbersome Plücker polynomials, the present approach has certain advantages: (1) The same methods work for $\bigwedge^m V$ and $V_1 \otimes \cdots \otimes V_m$, (2) the results have coordinate-free translations, and (3) the proofs are brief, as well as elementary.

The tensor space. Let V_i^* denote the space of linear functionals on V_i, and suppose $f_i \in V_i^*$, $i = 1, \ldots, m$. The universal factorization property of "$\otimes$" permits the definitions of unique linear transformations $c_j\colon V_1 \otimes \cdots \otimes V_m \to V_j$, $j = 1, \ldots, m$, and $c\colon V_1 \otimes \cdots \otimes V_m \to F$ satisfying

$$c_j(v_1 \otimes \cdots \otimes v_m) = \left[\prod_{\substack{i=1\\ i\neq j}}^{m} f_i(v_i)\right] v_j, \tag{1}$$

$$c(v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} f_i(v_i), \tag{2}$$

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for all $v_i \in V_i$, $i = 1, \ldots, m$. (The dependence of $c_1, \ldots, c_m, c$ on $f_1, \ldots, f_m$ will not be explicitly denoted.)

THEOREM 1. *A tensor $z \in V_1 \otimes \cdots \otimes V_m$ is decomposable if and only if for every choice of $f_i \in V_i^*$, $i = 1, \ldots, m$, we have*

$$c_1(z) \otimes \cdots \otimes c_m(z) = [c(z)]^{m-1} z, \tag{3}$$

where $c_1, \ldots, c_m, c$ are defined by (1), (2).

PROOF. Obvious from the definitions and the fact that if $z \neq 0$, then there exist $f_i \in V_i^*$, $i = 1, \ldots, m$, such that $c(z) \neq 0$. □

By choosing bases of the spaces $V_1, \ldots, V_m$, and by using the usual induced basis in $V_1 \otimes \cdots \otimes V_m$ and the dual bases in $V_1^*, \ldots, V_m^*$, we can convert Theorem 1 into a set of polynomial conditions on the coordinates of z which are equivalent to the decomposability of z.

Let V_i have the basis $v_{i1}, \ldots, v_{in_i}$ and let $f_{i1}, \ldots, f_{in_i}$ be the dual basis of V_i^*, $i = 1, \ldots, m$. The set

$$\{v_{1\alpha(1)} \otimes \cdots \otimes v_{m\alpha(m)} : 1 \leq \alpha(i) \leq n_i \text{ for } i = 1, \ldots, m\}$$

is a basis of $V_1 \otimes \cdots \otimes V_m$ [3] (notation: $v_{1\alpha(1)} \otimes \cdots \otimes v_{m\alpha(m)} = v_\alpha^\otimes$, $\Gamma = \{\alpha : 1 \leq \alpha(i) \leq n_i \text{ for } i = 1, \ldots, m\}$), so z has an expression

$$z = \sum_{\alpha \in \Gamma} a_\alpha v_\alpha^\otimes, \qquad a_\alpha \in F. \tag{4}$$

Now, if $f_i \in \{f_{i1}, \ldots, f_{in_i}\}$, $i = 1, \ldots, m$, say $f_i = f_{i\beta(i)}$, $\beta \in \Gamma$, then $c(z)$ and $c_j(z)$, $j = 1, \ldots, m$, may be computed using (1), (2), (4), and the duality of the v_{ij} and f_{ij} bases:

$$c(z) = a_\beta, \tag{5}$$

$$c_j(z) = \sum_{t=1}^{n_j} a_{\beta_{jt}} v_{jt}, \tag{6}$$

where $\beta_{jt} \in \Gamma$ is defined by

$$\beta_{jt}(i) = \begin{cases} \beta(i) & \text{if } i \neq j, \\ t & \text{if } i = j, \end{cases} \tag{7}$$

i.e., β_{jt} is the sequence obtained from β by replacing $\beta(j)$ by t. Compute each side of (3) using (5) and (6); (3) holds if and only if the coefficients of $v_\alpha^\otimes$ are equal for all $\alpha \in \Gamma$, i.e., if and only if

$$\prod_{i=1}^{m} a_{\beta_{i\alpha(i)}} - a_\beta^{m-1} a_\alpha = 0. \tag{8}$$

Thus, (8) holds for all $\alpha, \beta \in \Gamma$ if and only if (3) holds for every selection of $f_i \in \{f_{i1}, \ldots, f_{in_i}\}$, $i = 1, \ldots, m$, and by Theorem 1, this is equivalent to the decomposability of z. We have proved

THEOREM 2. *The tensor $z = \sum_{\alpha \in \Gamma} a_\alpha v_\alpha^\otimes \in V_1 \otimes \cdots \otimes V_m$ is decomposable if and only if* (8) *holds for all $\alpha, \beta \in \Gamma$.*

(Compare with the main result in [1].)

The Grassmann space. Let $f_i \in V^*$, $i = 1, \ldots, m$. The universal factorization property of "$\wedge$" permits the definitions of unique linear transformations $c_j\colon \bigwedge^m V \to V, j = 1, \ldots, m$, and $c\colon \bigwedge^m V \to F$ satisfying

$$c_j(v_1 \wedge \cdots \wedge v_m) \tag{9}$$
$$= \det \begin{bmatrix} f_1(v_1) & \cdots & f_{j-1}(v_1) & v_1 & f_{j+1}(v_1) & \cdots & f_m(v_1) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_1(v_m) & \cdots & f_{j-1}(v_m) & v_m & f_{j+1}(v_m) & \cdots & f_m(v_m) \end{bmatrix},$$

$$c(v_1 \wedge \cdots \wedge v_m) = \det \begin{bmatrix} f_1(v_1) & \cdots & f_m(v_1) \\ \vdots & \vdots & \vdots \\ f_1(v_m) & \cdots & f_m(v_m) \end{bmatrix}, \tag{10}$$

where the right side of (9) is interpreted as a vector in V, e.g.,

$$\det \begin{bmatrix} f_1(v_1) & v_1 & f_3(v_1) \\ f_1(v_2) & v_2 & f_3(v_2) \\ f_1(v_3) & v_3 & f_3(v_3) \end{bmatrix} = -\det \begin{bmatrix} f_1(v_2) & f_3(v_2) \\ f_1(v_3) & f_3(v_3) \end{bmatrix} v_1$$
$$+ \det \begin{bmatrix} f_1(v_1) & f_3(v_1) \\ f_1(v_3) & f_3(v_3) \end{bmatrix} v_2 - \det \begin{bmatrix} f_1(v_1) & f_3(v_1) \\ f_1(v_2) & f_3(v_2) \end{bmatrix} v_3.$$

THEOREM 3. *A skew-symmetric tensor $z \in \bigwedge^m V$ is decomposable if and only if for every choice of $f_i \in V^*$, $i = 1, \ldots, m$, we have*

$$c_1(z) \wedge \cdots \wedge c_m(z) = [c(z)]^{m-1} z, \tag{11}$$

where $c_1, \ldots, c_m, c$ are defined by (9), (10).

PROOF. The "if" part is obvious. For the converse, suppose $z = v_1 \wedge \cdots \wedge v_m$, and $f_1, \ldots, f_m \in V^*$ are given. Since each $c_j(z)$ is a linear combination of $v_1, \ldots, v_m$, $c_1(z) \wedge \cdots \wedge c_m(z)$ is a (possibly zero) multiple of $v_1 \wedge \cdots \wedge v_m$, say

$$c_1(z) \wedge \cdots \wedge c_m(z) = rz, \qquad r \in F. \tag{12}$$

From (9), it follows that

$$f_k(c_j(z)) = \begin{cases} c(z) & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

so applying c to both sides of (12) yields

$$[c(z)]^m = rc(z), \qquad c(z)\left(r - [c(z)]^{m-1}\right) = 0.$$

Thus, (11) holds when $c(z) \neq 0$. If $c(z) = 0$, then the adjugate of $[f_j(v_i)]$ is singular, and it follows easily that $c_1(z), \ldots, c_m(z)$ are linearly dependent, hence $c_1(z) \wedge \cdots \wedge c_m(z) = 0$, so (11) still holds, and the proof is complete.

By introducing a basis of V, a dual basis of V^*, and the usual induced basis in $\bigwedge^m V$ [3], one may proceed in a manner analogous to that in the previous section, translating the criterion (11) into a set of mth degree polynomial conditions on the coordinates of z w.r.t. the induced basis. We choose to omit this slightly messy but straightforward calculation.

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