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A NOTE ON k-CRITICALLY n-CONNECTED GRAPHS[1]

R. C. ENTRINGER AND PETER J. SLATER

ABSTRACT. A graph G is said to be (n^*, k)-connected if it has connectivity n and every set of k vertices is contained in an n-cutset. It is shown that an (n^*, k)-connected graph G contains an n-cutset C such that $G - C$ has a component with at most $n/(k+1)$ vertices, thereby generalizing a result of Chartrand, Kaugars and Lick. It is conjectured, however, that $n/(k+1)$ can be replaced with $n/2k$ and this is shown to be best possible.

The terminology and notation of [1] will be used throughout. In [4] the notion of a critically connected graph was generalized as follows. A graph with connectivity n is *k-critical* if whenever S is a vertex set with $|S| \leqslant k$ the connectivity of $G - S$ is $n - |S|$. That is, every vertex set with no more than k members is contained in an n-cutset or a trivializing set of n vertices. Such a graph will be said to be (n^*, k)-*connected.*

Chartrand, Kaugars, and Lick have shown [2] that an $(n^*, 1)$-connected graph, $n \geqslant 2$, contains a vertex of degree at most $3n/2 - 1$. This would follow, of course, if it were known that an $(n^*, 1)$-connected graph G contained an n-cutset C such that $G - C$ contained a component with at most $n/2$ vertices. Our object in this note is the generalization of the latter statement. To this end we develop the following notation.

Given a graph G with connectivity n let C_G be the family of all n-cutsets of G. If C is a member of C_G denote by $\nu(C)$ the number of vertices in a smallest component of $G - C$. Finally, let $r(G) = \min \nu(C)$ where the minimum is taken over all members C of C_G. That is, $r(G)$ is the order of a smallest component that can be obtained by removal of an n-cutset from G.

THEOREM. *If G is an (n^*, k)-connected graph with $1 \leqslant k < n$ then $r(G) \leqslant n/(k+1)$.*

PROOF. Choose an n-cutset C of G so that $\nu(C) = r \equiv r(G)$, and let R be the vertex set of a component of $G - C$ chosen so that $|R| = r$. Our proof will use the following property of R:

(i) An n-cutset D of G that contains a vertex of R contains all vertices of R.

To show this we let L be the vertex set of $G - C - R$, let T be the vertex set

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of one component of $G - D$ and let B be the vertex set of the remaining components of $G - D$ (see Figure 1).

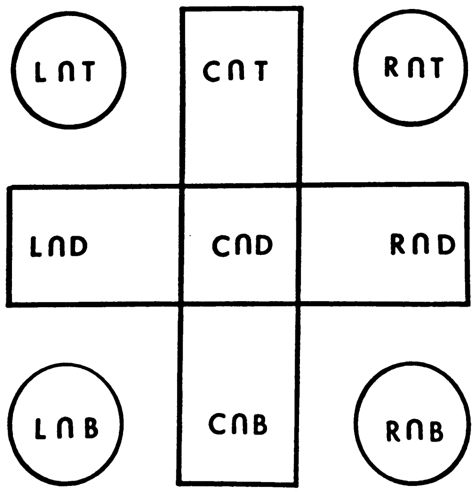

FIGURE 1

Since it is easy to argue that if $L \cap T$ is not empty then $(L \cap D) \cup (C \cap D) \cup (C \cap T)$ is a cutset of G, we suppress the details and note that analogous results hold for $L \cap B$, $R \cap T$, and $R \cap B$.

We can now conclude that at least one of the sets $L \cap T$ and $R \cap B$ is empty. If this were not so we must have $|(L \cap D) \cup (C \cap D) \cup (C \cap T)| \geqslant n$ and $|(R \cap D) \cup (C \cap D) \cup (C \cap B)| \geqslant n$. But equality must hold for both expressions since $|C| = |D| = n$. This is impossible, however, since it implies $r \leqslant |R \cap B| < |R|$. By similar argument one of the sets $L \cap B$ and $R \cap T$ must be empty.

Now, if (i) is not true, we may assume $R \cap T \neq \varnothing$ so that $L \cap B = \varnothing$. If, also, $R \cap B \neq \varnothing$ then $L \cap T = \varnothing$ so that $|R \cap D| < |R| \leqslant |L| = |L \cap D|$. Consequently, $|R \cap D| < \frac{1}{2}(n - |C \cap D|)$, which, in turn, gives

$$2n \leqslant |C \cap T| + 2|C \cap D| + 2|R \cap D| + |C \cap B| < |C| + n.$$

Since this is impossible we must have $R \cap B = \varnothing$ so that $|C \cap B| = |B| \geqslant |R| > |R \cap D|$. But this implies $(C \cap T) \cup (C \cap D) \cup (R \cap D)$ is a cutset with fewer than n vertices. Consequently (i) is proven and we now show:

(ii) The theorem holds for $k = 1$.

We assume otherwise and, referring to Figure 1, note that $|R \cap D| = |R| > n/2$ implies $|L \cap D| \leqslant |(L \cap D) \cup (C \cap D)| < n/2$, so that we assume $L \cap T \neq \varnothing$ and, consequently, must have $|C \cap T| > n/2$. This implies $|C \cap B| < n/2$ so that $L \cap B$ cannot be empty. But then $(L \cap D) \cup (C \cap D) \cup (C \cap B)$ is a cutset with fewer than n vertices, which cannot be. Hence (ii) holds and we now show:

(iii) $G - R$ is an $((n - r)^*, k - 1)$-connected graph.

$G - R$ is obviously $(n - r)$-connected. To show that any $k - 1$ vertices of $G - R$ lie in an $(n - r)$-cutset of $G - R$ we choose any such $(k - 1)$ set S of $G - R$ together with one vertex p of R and extend this to an n-cutset S' of G. By (i) R is a subset of S' so that $S' - R$ is an $(n - r)$-cutset of $G - R$ containing S, and (iii) is proven.

We can now complete the proof of the theorem by induction on k. We may assume it holds for all $(k - 1)$-critically connected graphs with $k > 1$. Then, by (iii), $G - R$ contains an $(n - r)$-cutset C' such that $G - R - C'$ has a component R' with at most $(n - r)/k$ vertices. But since $R \cup C'$ is an n-cutset of G we must have $r \leqslant (n - r)/k$, i.e., $r \leqslant n/(k + 1)$, and the proof is complete.

We obtain the following consequence immediately upon consideration of the degree of a vertex in the set R described in the proof of the theorem.

COROLLARY. *An (n^*, k)-connected graph contains a vertex of degree at most* $(k + 2)n/(k + 1) - 1$.

In particular, we can conclude that any (n^*, k)-connected graph with $k > (n - 1)/2$ has a vertex of degree n. In the cases $n = 2$ and 3 more is known, however. L. Nebesky [5] has shown that a (2, 1)-connected graph with at least six vertices contains four vertices of degree 2 and this result is best. The authors [3] have shown that a (3, 1)-connected graph contains at least two vertices of degree 3 and this result is best. Suppose, now, that G is a (4, 2)-connected graph. Then G has a vertex p of degree 4 so that, by (iii) above, $G - p$ is a (3, 1)-connected graph and consequently has two vertices of degree 3. G, then, had at least three vertices of degree 4. The possibility that such properties are not restricted to graphs with low connectivity can be made explicit as follows.

CONJECTURE 1. An (n, k)-connected graph with $k \geqslant (n - 1)/2$ contains at least two vertices of degree n.

We do not believe that the result of the theorem is best except at $k = 1$ and $k = [n/2]$, but, however, do have some confidence in the following conjecture.

CONJECTURE 2. If G is an (n^*, k)-connected graph then $r(G) \leqslant n/2k$.

We will describe a class of graphs, Figure 2, showing that this conjecture, if true, is best possible. For each $k \geqslant 1$ and $r \geqslant 1$ we define a graph $G_{k,r}$ as follows. The vertex set of $G_{k,r}$ consists of $2k + 2$ sets $S_1, \ldots, S_{2k+2}$ of r vertices each. Two vertices are adjacent if and only if they lie in sets S_i and S_j such that $i - j \not\equiv k + 1 \bmod(2k + 2)$. It is obvious that $G_{k,r}$ has connectivity $2kr$ and that $r(G_{k,r}) = r$. Also, for any choice of a set S of k vertices of $G_{k,r}$ there will be an i such that neither S_i or S_{i+k+1} (indices reduced mod$(2k + 1)$) contains a vertex of S. Consequently, S can be completed to a $2kr$-cutset and so $G_{k,r}$ is a $((2kr)^*, k)$-connected graph.

Conjecture 2, in addition to being correct, with proper interpretation, for noncritical graphs, i.e. at $k = 0$, would imply that if G is an (n, k)-connected

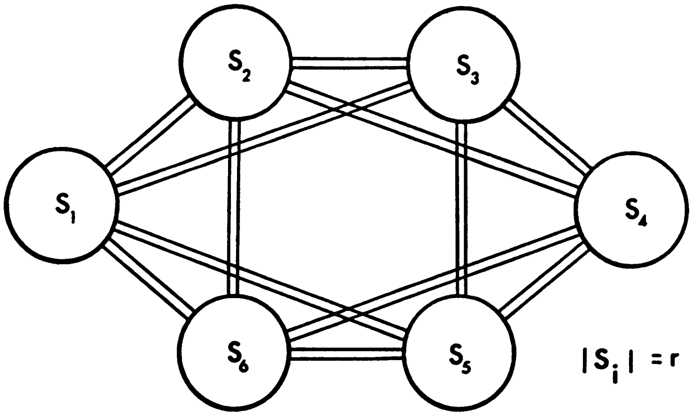

FIGURE 2

graph then either $k \leqslant [n/2]$ or $k = n$ and $G = k_{n+1}$. This latter implication has been conjectured by Slater [4].

NOTE ADDED IN PROOF. W. Mader has kindly informed us that property (i) in the proof of the Theorem had been previously proven by him [*Eine eigenschaft der atome endlicher graphen*, Arch. Math. (Basel) **22** (1971), 333–336.]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131

SANDIA LABORATORIES, ALBUQUERQUE, NEW MEXICO 87115