A NOTE ON $k$-CRITICALLY $n$-CONNECTED GRAPHS\textsuperscript{1}

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Abstract. A graph $G$ is said to be $(n^*, k)$-connected if it has connectivity $n$ and every set of $k$ vertices is contained in an $n$-cutset. It is shown that an $(n^*, k)$-connected graph $G$ contains an $n$-cutset $C$ such that $G - C$ has a component with at most $n/(k + 1)$ vertices, thereby generalizing a result of Chartrand, Kaugars and Lick. It is conjectured, however, that $n/(k + 1)$ can be replaced with $n/2k$ and this is shown to be best possible.

The terminology and notation of [1] will be used throughout. In [4] the notion of a critically connected graph was generalized as follows. A graph with connectivity $n$ is $k$-critical if whenever $S$ is a vertex set with $|S| < k$ the connectivity of $G - S$ is $n - |S|$. That is, every vertex set with no more than $k$ members is contained in an $n$-cutset or a trivializing set of $n$ vertices. Such a graph will be said to be $(n^*, k)$-connected.

Chartrand, Kaugars, and Lick have shown [2] that an $(n^*, 1)$-connected graph, $n > 2$, contains a vertex of degree at most $3n/2 - 1$. This would follow, of course, if it were known that an $(n^*, 1)$-connected graph $G$ contained an $n$-cutset $C$ such that $G - C$ contained a component with at most $n/2$ vertices. Our object in this note is the generalization of the latter statement. To this end we develop the following notation.

Given a graph $G$ with connectivity $n$ let $C_G$ be the family of all $n$-cutsets of $G$. If $C$ is a member of $C_G$ denote by $v(C)$ the number of vertices in a smallest component of $G - C$. Finally, let $r(G) = \min v(C)$ where the minimum is taken over all members $C$ of $C_G$. That is, $r(G)$ is the order of a smallest component that can be obtained by removal of an $n$-cutset from $G$.

**Theorem.** If $G$ is an $(n^*, k)$-connected graph with $1 < k < n$ then $r(G) < n/(k + 1)$.

**Proof.** Choose an $n$-cutset $C$ of $G$ so that $v(C) = r \equiv r(G)$, and let $R$ be the vertex set of a component of $G - C$ chosen so that $|R| = r$. Our proof will use the following property of $R$:

1. An $n$-cutset $D$ of $G$ that contains a vertex of $R$ contains all vertices of $R$.

To show this we let $L$ be the vertex set of $G - C - R$, let $T$ be the vertex set

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of one component of \( G - D \) and let \( B \) be the vertex set of the remaining components of \( G - D \) (see Figure 1).

\[
\begin{array}{ccc}
L \cap T & C \cap T & R \cap T \\
L \cap D & C \cap D & R \cap D \\
L \cap B & C \cap B & R \cap B
\end{array}
\]

\textbf{Figure 1}

Since it is easy to argue that if \( L \cap T \) is not empty then \((L \cap D) \cup (C \cap D) \cup (C \cap T)\) is a cutset of \( G \), we suppress the details and note that analogous results hold for \( L \cap B \), \( R \cap T \), and \( R \cap B \).

We can now conclude that at least one of the sets \( L \cap T \) and \( R \cap B \) is empty. If this were not so we must have \(|(L \cap D) \cup (C \cap D) \cup (C \cap T)| > n \) and \(|(R \cap D) \cup (C \cap D) \cup (C \cap B)| > n \). But equality must hold for both expressions since \(|C| = |D| = n\). This is impossible, however, since it implies \( r < |R \cap B| < |R| \). By similar argument one of the sets \( L \cap B \) and \( R \cap T \) must be empty.

Now, if (i) is not true, we may assume \( R \cap T \neq \emptyset \) so that \( L \cap B = \emptyset \). If, also, \( R \cap B \neq \emptyset \) then \( L \cap T = \emptyset \) so that \(|R \cap D| < |R| < |L| = |L \cap D|\). Consequently, \(|R \cap D| < \frac{1}{2}(n - |C \cap D|)\), which, in turn, gives

\[2n < |C \cap T| + 2|C \cap D| + 2|R \cap D| + |C \cap B| < |C| + n.\]

Since this is impossible we must have \( R \cap B = \emptyset \) so that \(|C \cap B| = |B| > |R| > |R \cap D|\). But this implies \((C \cap T) \cup (C \cap D) \cup (R \cap D)\) is a cutset with fewer than \( n \) vertices. Consequently (i) is proven and we now show:

(ii) The theorem holds for \( k = 1 \).

We assume otherwise and, referring to Figure 1, note that \(|R \cap D| = |R| > n/2 \) implies \(|L \cap D| < |(L \cap D) \cup (C \cap D)| < n/2 \), so that we assume \( L \cap T \neq \emptyset \) and, consequently, must have \(|C \cap T| > n/2 \). This implies \(|C \cap B| < n/2 \) so that \( L \cap B \) cannot be empty. But then \((L \cap D) \cup (C \cap D) \cup (C \cap B)\) is a cutset with fewer than \( n \) vertices, which cannot be. Hence (ii) holds and we now show:

(iii) \( G - R \) is an \(((n - r)^*, k - 1)\)-connected graph.
$G - R$ is obviously $(n - r)$-connected. To show that any $k - 1$ vertices of $G - R$ lie in an $(n - r)$-cutset of $G - R$ we choose any such $(k - 1)$ set $S$ of $G - R$ together with one vertex $p$ of $R$ and extend this to an $n$-cutset $S'$ of $G$. By (i) $R$ is a subset of $S'$ so that $S' - R$ is an $(n - r)$-cutset of $G - R$ containing $S$, and (iii) is proven.

We can now complete the proof of the theorem by induction on $k$. We may assume it holds for all $(k - 1)$-critically connected graphs with $k > 1$. Then, by (iii), $G - R$ contains an $(n - r)$-cutset $C'$ such that $G - R - C'$ has a component $R'$ with at most $(n - r)/k$ vertices. But since $R \cup C'$ is an $n$-cutset of $G$ we must have $r < (n - r)/k$, i.e., $r < n/(k + 1)$, and the proof is complete.

We obtain the following consequence immediately upon consideration of the degree of a vertex in the set $R$ described in the proof of the theorem.

**Corollary.** An $(n^*, k)$-connected graph contains a vertex of degree at most $(k + 2)n/(k + 1) - 1$.

In particular, we can conclude that any $(n^*, k)$-connected graph with $k > (n - 1)/2$ has a vertex of degree $n$. In the cases $n = 2$ and $3$ more is known, however. L. Nebesky [5] has shown that a $(2, 1)$-connected graph with at least six vertices contains four vertices of degree 2 and this result is best. The authors [3] have shown that a $(3, 1)$-connected graph contains at least two vertices of degree 3 and this result is best. Suppose, now, that $G$ is a $(4, 2)$-connected graph. Then $G$ has a vertex $p$ of degree 4 so that, by (iii) above, $G - p$ is a $(3, 1)$-connected graph and consequently has two vertices of degree 3. $G$, then, had at least three vertices of degree 4. The possibility that such properties are not restricted to graphs with low connectivity can be made explicit as follows.

**Conjecture 1.** An $(n, k)$-connected graph with $k \geq (n - 1)/2$ contains at least two vertices of degree $n$.

We do not believe that the result of the theorem is best except at $k = 1$ and $k = [n/2]$, but, however, do have some confidence in the following conjecture.

**Conjecture 2.** If $G$ is an $(n^*, k)$-connected graph then $r(G) < n/2k$.

We will describe a class of graphs, Figure 2, showing that this conjecture, if true, is best possible. For each $k > 1$ and $r > 1$ we define a graph $G_{k,r}$ as follows. The vertex set of $G_{k,r}$ consists of $2k + 2$ sets $S_1, \ldots, S_{2k+2}$ of $r$ vertices each. Two vertices are adjacent if and only if they lie in sets $S_i$ and $S_j$ such that $i - j \equiv k + 1 \mod(2k + 2)$. It is obvious that $G_{k,r}$ has connectivity $2kr$ and that $r(G_{k,r}) = r$. Also, for any choice of a set $S$ of $k$ vertices of $G_{k,r}$ there will be an $i$ such that neither $S_i$ or $S_{i+k+1}$ (indices reduced mod$(2k + 1)$) contains a vertex of $S$. Consequently, $S$ can be completed to a $2kr$-cutset and so $G_{k,r}$ is a $((2kr)^*, k)$-connected graph.

Conjecture 2, in addition to being correct, with proper interpretation, for noncritical graphs, i.e. at $k = 0$, would imply that if $G$ is an $(n, k)$-connected
Figure 2

A graph then either $k < \lfloor n/2 \rfloor$ or $k = n$ and $G = k_{n+1}$. This latter implication has been conjectured by Slater [4].

Note added in proof. W. Mader has kindly informed us that property (i) in the proof of the Theorem had been previously proven by him [	extit{Eine eigenschaft der atome endlicher graphen}, Arch. Math. (Basel) 22 (1971), 333–336.]

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