ON THE NORMABILITY OF THE INTERSECTION OF $L_p$ SPACES

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Abstract. The set $L_\omega = \bigcap_{p=1}^{\infty} L_p[0, 1]$ is not equal to $L_\infty[0, 1]$ since $L_\omega$ contains the function $-\ln x$. Using the theory of $L_p$ spaces for finitely additive set functions developed by Leader [9] we will prove several necessary and sufficient conditions for the normability of a generalization of $L_\omega$. These include the equality and finite dimensionality of all the $L_p$ spaces, $p > 1$.

1. Introduction. The set $L_\omega = \bigcap_{p=1}^{\infty} L_p[0, 1]$ is not equal to $L_\infty[0, 1]$ since $L_\omega$ contains the function $-\ln x$. In [2] Arens showed that with the natural $\omega$-topology, i.e. the smallest topology which contains each relative $p$-norm topology, $L_\omega$ is not normable. In [5] and [6] Davis, Murray and Weber discussed $L_p(\mu) = \bigcap_{q>p} L_q(\mu)$ which, when $p = \infty$ and $\mu$ is Lebesgue measure on $[0, 1]$, reduces to $L_\omega$. Using the theory of $L_p$ spaces for finitely additive set functions developed by Leader [9], we will prove several necessary and sufficient conditions for the normability of a generalization of $L_\omega$. These include the equality and finite dimensionality of all the $L_p$ spaces, $p > 1$.

Suppose $S$ is a set, $F$ a field of subsets of $S$ and $ba(F)$ the set of bounded finitely additive functions from $F$ into the real numbers. If $G \in ba(F)$, then $G^+$ will denote the nonnegative valued elements of $G$. $A_\lambda$ will be the set of elements of $ba(F)$ which are absolutely continuous with respect to $\lambda \in ba(F)^+$. It should be noted that we mean absolutely continuous in the $\epsilon-\delta$ sense which is (for $ba(F)$) stronger than the 0-0 sense. Henceforth, $\mu$ will be in $ba(F)^+$ with $\mu(S) = 1$. We adopt the convention that $a/0$ is 0.

2. "$L_p$ spaces" for $\mu$ and $H_\omega(\mu)$. For $1 < p < \infty$ we define $H_p(\mu)$ to be the set to which $\xi$ belongs iff

(1) $\xi \in A_\mu$ and

(2) there exists an $M > 0$ such that $\sum_D |\xi(I)|^p \mu(I)^{1-p} < M$ for each (finite) subdivision $D$ of $S$ by elements of $F$.

If $p > 1$, then (1) may be replaced by: $\mu(I) = 0$ implies $|\xi(I)| = 0$. If $p = 2$
we have the Hilbert space of Hellinger integrable functions which, when $F$ is a $\sigma$-field and $\mu$ is countably additive, corresponds to $L_2(\mu)$. The $p$-norm $(1 < p < \infty)$ denoted $\| \cdot \|_p$, is the $p$th root of the supremum of the sums in (2). $H_\infty(\mu)$ will be $\{ \xi \in A_\mu : |\xi(I)| \mu(I)^{-1} \text{ is bounded on } F \}$ (the Lip(\mu) of [4]) with $\| \xi \|_\infty = \sup(|\xi(I)| \mu(I)^{-1}) I \in F)$. With this norm $H_p(\mu) (1 < p < \infty)$ is a Banach space [9]. If $p = 1$, then $H_p(\mu)$ is $A_\mu$ with the total variation norm.

For $\xi, \eta$ and $\delta$ in $ba(F)$ we will denote by $\eta \wedge \delta$ the element of $ba(F)$ whose value at $v \in F$ is $\sup\{ g(v) | g \in ba(F), g \leq \eta \text{ and } g \leq \delta \}$, $\eta \vee \delta$ will be $-[(\eta) \wedge (-\delta)]$ and $\xi \vee 0$ will be written $\xi^+$, so that the total variation function for $\xi$ will be $|\xi| = \xi^+ + \xi^-$, where $\xi^-=(-\xi)^+$.

In the interest of self-containment we state some of the results from [9] which we will need.

**Theorem 2. L.** Suppose each of $\eta$ and $\delta$ is in $A_\mu$, $\lambda \in A_\mu^+$ and $1 < q < p < \infty$. Then we have:

I. If $|\delta| < |\eta|$, then $\eta \in H_p(\mu)$ iff $|\eta| \in H_p(\mu)$, in which case $\delta \in H_p(\mu)$ and $\|\delta\|_p < \|\eta\|_p = ||\eta||_p$.

II. The sequence $(\lambda \wedge K\mu)_{K=1}^\infty$ is $p$-norm convergent to $\lambda$ if $\lambda \in H_p(\mu)$ and, therefore, $H_\infty(\mu)$ is $p$-norm dense in $H_p(\mu)$.

III. The strength of the $p$-norms is nondecreasing with $p$ and therefore $H_\infty(\mu) \subseteq H_p(\mu) \subseteq H_q(\mu) \subseteq A_\mu$.

Suppose $\lambda \in A_\mu^+, 1 < p < \infty$, $D$ is a subdivision of $S$ and $N$ is the number of elements in $D$. Then there exists a $K \in \mathbb{N} = \{1, 2, \ldots \}$ such that $\lambda(v)^p < (\lambda \wedge K\mu)(v)^p + \mu(v)^{p-1}N^{-1}$ for each $v \in D$. Therefore

$$\sum_D \lambda(v)^p \mu(v)^{1-p} < \sum_D [(\lambda \wedge K\mu)(v)^p + \mu(v)^{p-1}N^{-1}] \mu(v)^{1-p}$$

$$< \|\lambda \wedge K\mu\|_P^p + 1,$$

and we have

**Lemma 2.1.** If $\lambda \in A_\mu^+$ and $1 < p < \infty$, then $\lambda \in H_p(\mu)$ iff $\{\|\lambda \wedge K\mu\|_P | K = 1, 2, \ldots \}$ is bounded.

We will denote by $H_\omega(\mu)$ the intersection of $H_p(\mu)$ for $1 < p < \infty$. The $\omega$-topology will be the smallest topology on $H_\omega(\mu)$ which contains each relative $p$-norm topology.

**Lemma 2.2.** If $(a_p)_{p=1}^\infty$ is a sequence of positive numbers such that $\sum_{p=1}^\infty a_p \|\xi\|_p$ exists for each $\xi$ in $H_\omega(\mu)$, then the norm defined by $\|\xi\| = \sum_{p=1}^\infty a_p \|\xi\|_p$ is complete.

**Proof.** Let $(\xi_n)$ be a $\| \cdot \|$ Cauchy sequence in $H_\omega(\mu)$. Then by 2.L.II we have

$$\| |\xi_n| - |\xi_m| \| = || |\xi_n| - |\xi_m| | \| \leq || |\xi_n - \xi_m| \| = \| \xi_n - \xi_m \|$$

so that $(|\xi_n|)$ is also $\| \cdot \|$ Cauchy. Let $(\delta_n) = (|\xi_n|)$ be a subsequence of $(|\xi_n|)$
such that if $i$ and $r$ are positive integers then $\|\delta_i - \delta_{i+r}\| < 1/2^r$. Let $\eta_1 = \delta_1$ and for each $1 < i \in \mathbb{N}$ let $\eta_i = \delta_i \vee \eta_{i-1}$. Then

$0 < \eta_{i+1} - \eta_i = (\delta_{i+1} - \eta_i)^+ < (\delta_{i+1} - \delta_i)^+ < |\delta_{i+1} - \delta_i|$, so that again by 2.L.II, $\|\eta_{i+1} - \eta_i\| < \|\delta_{i+1} - \delta_i\| < 1/2^r$. Now for each $i$, $r \in \mathbb{N}$ we have

$$\|\eta_{i+r} - \eta_i\| < \sum_{j=1}^{r} \|\eta_{i+j} - \eta_{i+j-1}\| < \sum_{j=1}^{r} \frac{1}{2^j} < \frac{1}{2^{r-1}}$$

and, therefore, $(\eta_i)$ is $\|\cdot\|$ Cauchy and $\eta_i < \eta_{i+1}$ for each $i \in \mathbb{N}$.

Now any norm Cauchy sequence is $p$-norm Cauchy for each $1 < p < \infty$ and, hence, has a $p$-norm limit which (by the comparability of $p$-norms) is independent of $p$. Therefore there exists $\xi$ and $\eta$ in $H^p(\mu)$ such that $\eta_i \to^p \eta$ and $\xi_i \to^p \xi$ for each $1 < p < \infty$.

Now let $c > 0$ and $K \in \mathbb{N}$ be such that $\sum_{K+1}^\infty a_p\|\eta\|_p < c/4$ and $\sum_{K+1}^\infty a_p\|\xi\|_p < c/4$. For each $p < K$ let $N_p$ be such that if $i > N_p$, then $a_p\|\xi - \xi_n\|_p < c/2K$. Let $N = \max\{N_1, N_2, \ldots, N_K\}$ and $i > N$. Then

$$\|\xi - \xi_n\| = \sum_{p=1}^\infty a_p\|\xi - \xi_n\|_p
= \sum_{p=1}^K a_p\|\xi - \xi_n\|_p + \sum_{K+1}^\infty a_p\|\xi - \xi_n\|_p
\leq \sum_{p=1}^K \frac{c}{2K} + \sum_{K+1}^\infty a_p(\|\xi\|_p + \|\xi_n\|_p)
\leq \frac{c}{2} + \frac{c}{4} + \sum_{K+1}^\infty a_p\|\delta_i\|_p
\leq \frac{3c}{4} + \sum_{K+1}^\infty a_p\|\eta_i\|_p
\leq \frac{3c}{4} + \sum_{K+1}^\infty a_p\|\eta\|_p < c.$$  

Therefore, $\xi_n \to^{\|\cdot\|} \xi$ and, hence, $\xi_n \to^{\|\cdot\|} \xi$, so that $\|\cdot\|$ is complete.

3. A differential equivalence theorem for Hellinger integrals. In this section we will show that $H_{2^k}(\mu)$ is the “$K$th image” of a certain function. The integral considered below is the refinement limit of sums over finite subdivisions of $S$ by elements of $F$. For further details see [1].

**Theorem 3.H.** If $\xi \in ba(F)$, then $\xi \in A^+_\mu$ iff there exists an $\eta \in H_2(\mu)$ such that $\xi = \int \eta^2/\mu$.

The proof of this theorem of Hellinger [8] (for interval functions) carries over to our setting and suggests the function

$$T: H^+_2(\mu) \to A^+_\mu: \eta \to \int \eta^2/\mu.$$
The $\eta$ of Theorem 3.H is given by the function $R(\xi) = f(\xi, \mu)^{1/2}$ which has
the following properties [3]. $R$ is defined on all of $ba(F)^+$ and its restriction
to $A_\mu^+$ is the inverse of $T$. If $\xi \in ba(F)^+$, then $R(\xi) \in H_2^+(\mu)$ and $T(R(\xi)) = a_\mu(\xi)$ (the absolutely continuous part of $\xi$).

**Lemma 3.1.** Suppose $\alpha: F \to \mathbb{R}^+$, $\mu(I) = 0$ implies $\alpha(I) = 0$ for each $I \in F$, $\Sigma_p \alpha(I)$ is nondecreasing for successive refinements and $\int \alpha(I)$ exists. Then $\int \alpha^2/\mu$ exists iff $\int (\alpha)^2/\mu$ exists, in which case they are equal.

**Proof.** If $\int (\alpha)^2/\mu$ exists, then, since $\Sigma_p \alpha^2(I)/\mu(I)$ is nondecreasing for successive refinements and bounded by $\int (\alpha(I))^2/\mu(v)$, we have $\int \alpha^2/\mu$ exists.

If $\int \alpha^2/\mu$ exists, then

$$\int \alpha = \left( \mu \int \alpha^2/\mu \right)^{1/2} = \left( \mu \int \alpha^2/\mu \right)^{1/2}$$

(see [1])

$$= R \left( \int \alpha^2/\mu \right) \in H_2(\mu),$$

i.e. $\int (\alpha)^2/\mu$ exists.

If both exist, then

$$\int \alpha^2/\mu \leq \int \left( \int \alpha \right)^2/\mu = T(\int \alpha) = T \left( R \left( \int \alpha^2/\mu \right) \right) = a_\mu \left( \int \alpha^2/\mu \right) \leq \int \alpha^2/\mu.$$

For $\xi \in ba(F)$ and $p > 1$ we have $\Sigma_p \|\xi(I)\|^p \mu(I) \leq p$ is nondecreasing for successive refinements [9] so that, for $\xi \in A_\mu^+$, $\xi \in H_p(\mu)$ iff $\int \|\xi\|^p \mu^{1-p}$ exists. Therefore by 3.1 and induction we have, for $\xi \in A_\mu^+$, $T^K(\xi)$ exists iff $\xi \in H^p(\mu)$. Hence $R^K(A_\mu^+) = H^p(\mu)$.

4. The normability of $H^\omega(\mu)$. In the countably additive case the equivalence of (2) and (3) below can be found in [5].

**Theorem 4.1.** These are equivalent:

1. The $\omega$-topology is contained in a norm topology.
2. The $\omega$-topology is normable.
3. There exists a $p (1 < p < \infty)$ such that $H_p(\mu) = H^\omega(\mu)$.
4. $A_\mu = H_2(\mu)$.
5. $A_\mu = H_\infty(\mu)$.
6. $A_\mu$ is finite dimensional.

(1) $\to$ (2). Suppose $\| \cdot \|$ is a norm on $H^\omega(\mu)$ and its topology contains the $\omega$-topology. Then for each $p \in \mathbb{N}$ there exists an $M_p > 0$ such that $\|\xi\|_p < M_p \|\xi\|$ for each $\xi \in H^\omega(\mu)$ and, therefore, $\| \cdot \| = \sum_{p=1}^\infty \| \cdot \|_p 2^{-p} M_p^{-1}$ is a complete norm by Lemma 2.2. The $\omega$-topology, being defined by a countable collection of norms, is a complete linear metric topology which is contained in the $\| \cdot \|$ topology and, hence, equal to it by the open mapping theorem [7].

(2) $\to$ (3). Suppose the $\omega$-topology is normable, then there exists a neighborhood $U$ of 0 which is $\omega$-bounded in the following sense: if $(\eta_n) \subseteq U$ and
(\alpha_n) is a null sequence, then (\alpha_n \eta_n \to \omega 0). Since the \omega-topology is generated by the p-norms, \rho \in N, there exists a p > 1 such that if B = \{ \eta \in H, (\mu) \mid \|\eta\|_p < 1\}, then B \cap H, (\mu) is \omega-bounded. Now let \xi \in B^+, r > p and \\
\xi_k = \xi / K\mu for each K \in N. Then \|\xi_k\|_r is bounded, since otherwise we would have \((1/\|\xi_k\|_r)\|^{1/2} \to 0\) and, therefore, \((1/\|\xi_k\|_r)^{1/2} \xi_k \omega 0\), hence \((1/\|\xi_k\|_r)^{1/2} \xi_k \to 0\), which is a contradiction. Therefore, by Lemma 2.1, \xi \in H, (\mu) and, hence, H, (\mu) = H, (\mu).

(3) \to (4). Suppose p > 1 and H, (\mu) = H, (\mu). By 2.L. it is sufficient to consider A, +. Let K \in N be such that p < 2K, then by \S3 we have

\[ A^+_\mu = T^K(R^K(A^+_\mu)) = T^K(H^+_2(\mu)) = T^K(H^+_2(\mu)) \]

We will need the following to prove that (4) implies (5).

**Theorem 4.B.** If each of \lambda and \mu is in ba(F)*, then \lambda \in H_\infty (\mu) iff H^+_2(\lambda) \subseteq H_2(\mu).

**Proof.** [4].

(4) \to (5). Suppose H_2(\mu) = A_\mu and \lambda \in A^+_\mu. Then H_2(\lambda) \subseteq A_\lambda \subseteq A_\mu = H_2(\mu) and, therefore, by 4.B, \lambda \in H_\infty (\mu).

(5) \to (6). Suppose A_\mu is not finite dimensional; then there exists a disjoint sequence, (v_n), on which \mu is positive. Let (a_n) be an unbounded sequence of positive numbers such that \sum_{n=1}^\infty a_n \mu(v_n) < \infty. Then \sum_{n=1}^\infty a_n \mu_n is in A_\mu but not H_\infty (\mu), where \mu_n is the contraction of \mu to v_n, i.e. \mu_n(I) = \mu(I \cap v_n) for each I \in F.

(6) \to (1). This follows from the fact that \| \cdot \|_\infty is stronger than each relative p-norm.

We conclude by noting that as a consequence of the equivalence of (1), (2) and (5) we have the following “internal” characterization of the normability of H, (\mu).

**Corollary 4.1.** The \omega-topology is normable iff H, (\mu) = H_\infty (\mu).

**Added in proof.** For countably additive \mu on a \sigma-field the property H, (\mu) = H_\infty (\mu), 4.1.5, 4.1.6 and several other conditions have been considered by Professor Ion Chitescu in Finitely purely atomic measures and L^p-spaces, Anal. Univ. București Sti. Natur. 24 (1975), 23–29, MR 52 #8366.

**Bibliography**


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