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ON THE NORMABILITY OF THE INTERSECTION OF L_p SPACES[1]

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ABSTRACT. The set $L_\omega = \bigcap_{p=1}^{\infty} L_p[0, 1]$ is not equal to $L_\infty[0, 1]$ since L_ω contains the function $-\ln x$. Using the theory of L_p spaces for finitely additive set functions developed by Leader [9] we will prove several necessary and sufficient conditions for the normability of a generalization of L_ω. These include the equality and finite dimensionality of all the L_p spaces, $p > 1$.

1. Introduction. The set $L_\omega = \bigcap_{p=1}^{\infty} L_p[0, 1]$ is not equal to $L_\infty[0, 1]$ since L_ω contains the function $-\ln x$. In [2] Arens showed that with the natural ω-topology, i.e. the smallest topology which contains each relative p-norm topology, L_ω is not normable. In [5] and [6] Davis, Murray and Weber discussed $L_{p^-}(\mu) = \bigcap_{q<p} L_q(\mu)$ which, when $p = \infty$ and μ is Lebesgue measure on [0, 1], reduces to L_ω. Using the theory of L_p spaces for finitely additive set functions developed by Leader [9], we will prove several necessary and sufficient conditions for the normability of a generalization of L_ω. These include the equality and finite dimensionality of all the L_p spaces, $p \geqslant 1$.

Suppose S is a set, F a field of subsets of S and $\mathrm{ba}(F)$ the set of bounded finitely additive functions from F into the real numbers. If $G \subseteq \mathrm{ba}(F)$, then G^+ will denote the nonnegative valued elements of G. A_λ will be the set of elements of $\mathrm{ba}(F)$ which are absolutely continuous with respect to $\lambda \in \mathrm{ba}(F)^+$. It should be noted that we mean absolutely continuous in the ε-δ sense which is (for $\mathrm{ba}(F)$) stronger than the 0-0 sense. Henceforth, μ will be in $\mathrm{ba}(F)^+$ with $\mu(S) = 1$. We adopt the convention that $a/0$ is 0.

2. "L_p spaces" for μ and $H_\omega(\mu)$. For $1 \leqslant p < \infty$ we define $H_p(\mu)$ to be the set to which ξ belongs iff

(1) $\xi \in A_\mu$ and

(2) there exists an $M > 0$ such that $\sum_D |\xi(I)|^p \mu(I)^{1-p} < M$ for each (finite) subdivision D of S by elements of F.

If $p > 1$, then (1) may be replaced by: $\mu(I) = 0$ implies $|\xi(I)| = 0$. If $p = 2$

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we have the Hilbert space of Hellinger integrable functions which, when F is a σ-field and μ is countably additive, corresponds to $L_2(\mu)$. The p-norm ($1 \leqslant p < \infty$) denoted, $\|\cdot\|_p$, is the pth root of the supremum of the sums in (2). $H_\infty(\mu)$ will be $\{\xi \in A_\mu \mid |\xi(I)|\,\mu(I)^{-1}$ is bounded on $F\}$ (the Lip(μ) of [4]) with $\|\xi\|_\infty = \sup\{|\xi(I)|\,\mu(I)^{-1} \mid I \in F\}$. With this norm $H_p(\mu)$ ($1 \leqslant p \leqslant \infty$) is a Banach space [9]. If $p = 1$, then $H_p(\mu)$ is A_μ with the total variation norm.

For ξ, η and δ in ba(F) we will denote by $\eta \wedge \delta$ the element of ba(F) whose value at $v \in F$ is $\sup\{g(v) \mid g \in \mathrm{ba}(F),\ g \leqslant \eta$ and $g \leqslant \delta\}$, $\eta \vee \delta$ will be $-[(-\eta) \wedge (-\delta)]$ and $\xi \vee 0$ will be written ξ^+, so that the total variation function for ξ will be $|\xi| = \xi^+ + \xi^-$, where $\xi^- = (-\xi)^+$.

In the interest of self-containment we state some of the results from [9] which we will need.

THEOREM 2. L. *Suppose each of η and δ is in A_μ, $\lambda \in A_\mu^+$ and $1 \leqslant q \leqslant p \leqslant \infty$. Then we have;*

I. *If $|\delta| \leqslant |\eta|$, then $\eta \in H_p(\mu)$ iff $|\eta| \in H_p(\mu)$, in which case $\delta \in H_p(\mu)$ and $\|\delta\|_p \leqslant \|\eta\|_p = \|\,|\eta|\,\|_p$.*

II. *The sequence $(\lambda \wedge K\mu)_{K=1}^\infty$ is p-norm convergent to λ iff $\lambda \in H_p(\mu)$ and, therefore, $H_\infty(\mu)$ is p-norm dense in $H_p(\mu)$.*

III. *The strength of the p-norms is nondecreasing with p and therefore*

$$H_\infty(\mu) \subseteq H_p(\mu) \subseteq H_q(\mu) \subseteq A_\mu.$$

Suppose $\lambda \in A_\mu^+$, $1 < p < \infty$, D is a subdivision of S and N is the number of elements in D. Then there exists a $K \in \mathbf{N} = \{1, 2, \ldots\}$ such that $\lambda(v)^p \leqslant (\lambda \wedge K\mu)(v)^p + \mu(v)^{p-1}N^{-1}$ for each $v \in D$. Therefore

$$\sum_D \lambda(v)^p \mu(v)^{1-p} \leqslant \sum_D \left[(\lambda \wedge K\mu)(v)^p + \mu(v)^{p-1}N^{-1}\right]\mu(v)^{1-p}$$
$$\leqslant \|\lambda \wedge K\mu\|_p^p + 1,$$

and we have

LEMMA 2.1. *If $\lambda \in A_\mu^+$ and $1 \leqslant p \leqslant \infty$, then $\lambda \in H_p(\mu)$ iff $\{\|\lambda \wedge K\mu\|_p \mid K = 1, 2, \ldots\}$ is bounded.*

We will denote by $H_\omega(\mu)$ the intersection of $H_p(\mu)$ for $1 \leqslant p < \infty$. The ω-topology will be the smallest topology on $H_\omega(\mu)$ which contains each relative p-norm topology.

LEMMA 2.2. *If $(a_p)_{p=1}^\infty$ is a sequence of positive numbers such that $\sum_{p=1}^\infty a_p\|\xi\|_p$ exists for each ξ in $H_\omega(\mu)$, then the norm defined by $\|\xi\| = \sum_{p=1}^\infty a_p\|\xi\|_p$ is complete.*

PROOF. Let (ξ_n) be a $\|\cdot\|$ Cauchy sequence in $H_\omega(\mu)$. Then by 2.L.II we have

$$\|\,|\xi_n| - |\xi_m|\,\| = \|\,|\,|\xi_n| - |\xi_m|\,|\,\| \leqslant \|\,|\xi_n - \xi_m|\,\| = \|\xi_n - \xi_m\|$$

so that $(|\xi_n|)$ is also $\|\cdot\|$ Cauchy. Let $(\delta_i) = (|\xi_{n_i}|)$ be a subsequence of $(|\xi_n|)$

such that if i and r are positive integers then $\|\delta_i - \delta_{i+r}\| < 1/2^i$. Let $\eta_1 = \delta_1$ and for each $1 < i \in \mathbf{N}$ let $\eta_i = \delta_i \vee \eta_{i-1}$. Then

$$0 \leqslant \eta_{i+1} - \eta_i = (\delta_{i+1} - \eta_i)^+ \leqslant (\delta_{i+1} - \delta_i)^+ \leqslant |\delta_{i+1} - \delta_i|,$$

so that again by 2.L.II, $\|\eta_{i+1} - \eta_i\| \leqslant \|\delta_{i+1} - \delta_i\| < 1/2^i$. Now for each i, $r \in \mathbf{N}$ we have

$$\|\eta_{i+r} - \eta_i\| \leqslant \sum_{j=1}^{r} \|\eta_{i+j} - \eta_{i+j-1}\| < \sum_{j=1}^{r} \frac{1}{2^{i+j-1}} < \frac{1}{2^{i-1}}$$

and, therefore, (η_i) is $\|\cdot\|$ Cauchy and $\eta_i \leqslant \eta_{i+1}$ for each $i \in \mathbf{N}$.

Now any norm Cauchy sequence is p-norm Cauchy for each $1 \leqslant p < \infty$ and, hence, has a p-norm limit which (by the comparability of p-norms) is independent of p. Therefore there exists ξ and η in $H_\omega(\mu)$ such that $\eta_i \to^p \eta$ and $\xi_n \to^p \xi$ for each $1 \leqslant p < \infty$.

Now let $c > 0$ and $K \in \mathbf{N}$ be such that $\sum_{K+1}^{\infty} a_p\|\eta\|_p < c/4$ and $\sum_{K+1}^{\infty} a_p\|\xi\|_p < c/4$. For each $p \leqslant K$ let N_p be such that if $i > N_p$, then $a_p\|\xi - \xi_{n_i}\|_p < c/2K$. Let $N = \max\{N_1, N_2, \ldots, N_K\}$ and $i > N$. Then

$$\begin{aligned}
\|\xi - \xi_{n_i}\| &= \sum_{p=1}^{\infty} a_p\|\xi - \xi_{n_i}\|_p \\
&= \sum_{p=1}^{K} a_p\|\xi - \xi_{n_i}\|_p + \sum_{K+1}^{\infty} a_p\|\xi - \xi_{n_i}\|_p \\
&\leqslant \sum_{p=1}^{K} \frac{c}{2K} + \sum_{K+1}^{\infty} a_p\left(\|\xi\|_p + \|\xi_{n_i}\|_p\right) \\
&\leqslant \frac{c}{2} + \frac{c}{4} + \sum_{K+1}^{\infty} a_p\|\delta_i\|_p \\
&\leqslant \frac{3c}{4} + \sum_{K+1}^{\infty} a_p\|\eta_i\|_p \\
&\leqslant \frac{3c}{4} + \sum_{K+1}^{\infty} a_p\|\eta\|_p < c.
\end{aligned}$$

Therefore, $\xi_{n_i} \to^{\|\ \|} \xi$ and, hence, $\xi_n \to^{\|\ \|} \xi$, so that $\|\ \|$ is complete.

3. A differential equivalence theorem for Hellinger integrals. In this section we will show that $H_{2^K}(\mu)$ is the "Kth image" of a certain function. The integral considered below is the refinement limit of sums over finite subdivisions of S by elements of F. For further details see **[1]**.

THEOREM 3.H. *If $\xi \in \mathrm{ba}(F)$, then $\xi \in A_\mu^+$ iff there exists an $\eta \in H_2(\mu)$ such that $\xi = \int \eta^2/\mu$.*

The proof of this theorem of Hellinger **[8]** (for interval functions) carries over to our setting and suggests the function

$$T\colon H_2^+(\mu) \to A_\mu^+ \colon \eta \to \int \eta^2/\mu.$$

The η of Theorem 3.H is given by the function $R(\xi) = \int(\xi, \mu)^{1/2}$ which has the following properties [3]. R is defined on all of $\mathrm{ba}(F)^+$ and its restriction to A_μ^+ is the inverse of T. If $\xi \in \mathrm{ba}(F)^+$, then $R(\xi) \in H_2^+(\mu)$ and $T(R(\xi)) = \mathbf{a}_\mu(\xi)$ (the absolutely continuous part of ξ).

LEMMA 3.1. *Suppose α: $F \to \mathbf{R}^+$, $\mu(I) = 0$ implies $\alpha(I) = 0$ for each $I \in F$, $\Sigma_D \alpha(I)$ is nondecreasing for successive refinements and $\int_S \alpha(I)$ exists. Then $\int \alpha^2/\mu$ exists iff $\int(\int \alpha)^2/\mu$ exists, in which case they are equal.*

PROOF. If $\int(\int \alpha)^2/\mu$ exists, then, since $\Sigma_D\, \alpha^2(I)/\mu(I)$ is nondecreasing for successive refinements and bounded by $\int_S(\int_v \alpha(I))^2/\mu(v)$, we have $\int \alpha^2/\mu$ exists.

If $\int \alpha^2/\mu$ exists, then

$$\int \alpha = \int (\mu\alpha^2/\mu)^{1/2} = \int \left(\mu \int \alpha^2/\mu\right)^{1/2} \quad (\text{see } [\mathbf{1}])$$

$$= R\left(\int \alpha^2/\mu\right) \in H_2(\mu),$$

i.e. $\int(\int \alpha)^2/\mu$ exists.

If both exist, then

$$\int \alpha^2/\mu \leqslant \int \left(\int \alpha\right)^2/\mu = T\left(\int \alpha\right) = T\left(R\left(\int \alpha^2/\mu\right)\right) = \mathbf{a}_\mu\left(\int \alpha^2/\mu\right) \leqslant \int \alpha^2/\mu.$$

For $\xi \in \mathrm{ba}(F)$ and $p \geqslant 1$ we have $\Sigma_D |\xi(I)|^p \mu(I)^{1-p}$ is nondecreasing for successive refinements [9] so that, for $\xi \in A_\mu$, $\xi \in H_p(\mu)$ iff $\int |\xi|^p \mu^{1-p}$ exists. Therefore by 3.1 and induction we have, for $\xi \in A_\mu^+$, $T^K(\xi)$ exists iff $\xi \in H_{2^K}(\mu)$. Hence $R^K(A_\mu^+) = H_{2^K}^+(\mu)$.

4. The normability of $H_\omega(\mu)$. In the countably additive case the equivalence of (2) and (3) below can be found in [5].

THEOREM 4.1. *These are equivalent*:

(1) *The ω-topology is contained in a norm topology.*

(2) *The ω-topology is normable.*

(3) *There exists a p $(1 \leqslant p < \infty)$ such that $H_p(\mu) = H_\omega(\mu)$.*

(4) $A_\mu = H_2(\mu)$.

(5) $A_\mu = H_\infty(\mu)$.

(6) *A_μ is finite dimensional.*

(1) $\to$ (2). Suppose $|||\cdot|||$ is a norm on $H_\omega(\mu)$ and its topology contains the ω-topology. Then for each $p \in \mathbf{N}$ there exists an $M_p > 0$ such that $\|\xi\|_p \leqslant M_p |||\xi|||$ for each $\xi \in H_\omega(\mu)$ and, therefore, $\|\cdot\| = \Sigma_{p=1}^\infty \|\cdot\|_p 2^{-p} M_p^{-1}$ is a complete norm by Lemma 2.2. The ω-topology, being defined by a countable collection of norms, is a complete linear metric topology which is contained in the $\|\cdot\|$ topology and, hence, equal to it by the open mapping theorem [7].

(2) $\to$ (3). Suppose the ω-topology is normable, then there exists a neighborhood U of 0 which is ω-bounded in the following sense: if $(\eta_n) \subseteq U$ and

(α_n) is a null sequence, then $(\alpha_n \eta_n \to^{\omega} 0)$. Since the ω-topology is generated by the p-norms, $p \in \mathbf{N}$, there exists a $p \geqslant 1$ such that if $B = \{\eta \in H_p(\mu) | \|\eta\|_p \leqslant 1\}$, then $B \cap H_\omega(\mu)$ is ω-bounded. Now let $\xi \in B^+$, $r > p$ and $\xi_K = \xi \wedge K\mu$ for each $K \in \mathbf{N}$. Then $\|\xi_K\|_r$ is bounded, since otherwise we would have $(1/\|\xi_K\|_r)^{1/2} \to 0$ and, therefore, $(1/\|\xi_K\|_r)^{1/2}\xi_K \to^{\omega} 0$, hence $(1/\|\xi_K\|_r)^{1/2}\, \xi_K$ r-norm convergent to 0, which is a contradiction. Therefore, by Lemma 2.1, $\xi \in H_r(\mu)$ and, hence, $H_p(\mu) = H_\omega(\mu)$.

(3) $\to$ (4). Suppose $p \geqslant 1$ and $H_\omega(\mu) = H_p(\mu)$. By 2.L. it is sufficient to consider A_μ^+. Let $K \in \mathbf{N}$ be such that $p < 2^K$, then by §3 we have

$$A_\mu^+ = T^K\left(R^K(A_\mu^+)\right) = T^K\left(H_{2^K}^+(\mu)\right) = T^K\left(H_{2^{K+1}}^+(\mu)\right)$$
$$= T^K\left(R^{K+1}(A_\mu^+)\right) = R(A_\mu^+) = H_2^+(\mu).$$

We will need the following to prove that (4) implies (5).

THEOREM 4.B. *If each of* λ *and* μ *is in* $\mathrm{ba}(F)^+$, *then* $\lambda \in H_\infty(\mu)$ *iff* $H_2^+(\lambda) \subseteq H_2(\mu)$.

PROOF. [4].

(4) $\to$ (5). Suppose $H_2(\mu) = A_\mu$ and $\lambda \in A_\mu^+$. Then $H_2(\lambda) \subseteq A_\lambda \subseteq A_\mu = H_2(\mu)$ and, therefore, by 4.B, $\lambda \in H_\infty(\mu)$.

(5) $\to$ (6). Suppose A_μ is not finite dimensional; then there exists a disjoint sequence, (v_n), on which μ is positive. Let (a_n) be an unbounded sequence of positive numbers such that $\sum_{n=1}^\infty a_n\, \mu(v_n) < \infty$. Then $\sum_{n=1}^\infty a_n\, \mu_n$ is in A_μ but not $H_\infty(\mu)$, where μ_n is the contraction of μ to v_n, i.e. $\mu_n(I) = \mu(I \cap v_n)$ for each $I \in F$.

(6) $\to$ (1). This follows from the fact that $\|\cdot\|_\infty$ is stronger than each relative p-norm.

We conclude by noting that as a consequence of the equivalence of (1), (2) and (5) we have the following "internal" characterization of the normability of $H_\omega(\mu)$.

COROLLARY 4.1. *The* ω-*topology is normable iff* $H_\omega(\mu) = H_\infty(\mu)$.

ADDED IN PROOF. For countably additive μ on a σ-field the property $H_\omega(\mu) = H_\infty(\mu)$, 4.1.5, 4.1.6 and several other conditions have been considered by Professor Ion Chitescu in *Finitely purely atomic measures and L^p-spaces*, Anal. Univ. Bucureşti Sti. Natur. **24** (1975), 23–29, MR **52** #8366.

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