PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 67, Number 1, November 1977

# THE COMPLETION OF PRÜFER DOMAINS

JIŘÍ MOČKOŘ

ABSTRACT. The completion of Prüfer and almost Dedekind domains in topologies $\sup\{\mathfrak{T}_w: w \in \Omega\}$, where $\mathfrak{T}_w$ is a topology induced by a valuation $w \in \Omega$, are characterized in terms of Manis valuations.

**1.** Let $A$ be an integral domain with the quotient field $K$ and let $\Omega$ be the family of nontrivial valuations on $K$ which are nonnegative on $A$. It is well known that a valuation $w \in \Omega$ with the value group $G_w$ defines a field topology $\mathfrak{T}_w$ in $K$ with the sets $U_{w,\alpha} = \{x \in K: w(x) > \alpha\}$, $\alpha \in G_w^+ = \{\beta \in G_w: \beta \geqslant 0\}$, as a base of the neighbourhoods of zero in $K$. Since $\mathfrak{T}_w$ is a minimal topology in the ordered set of all field topologies on $K$, the completion $\hat{K}_w$ of $K$ is a field and the extension $\hat{w}$ of $w$ on $\hat{K}_w$ is a valuation on $\hat{K}_w$.

In this note we consider a more general situation. Let $\mathfrak{T}$ be the supremum of the family of topologies $\{\mathfrak{T}_w: w \in \Omega\}$. In general, the family $\Omega$ contains nonequivalent valuations, and, in this case, the completion $\hat{K}$ of $K$ in $\mathfrak{T}$ may contain zero divisors. Hence, it seems natural to use for the investigation of the ideal-theoretic properties of the completion $\hat{A}$ of a domain $A$ in a topology $\mathfrak{T}$ the valuation theory on rings with zero divisors which was introduced by M. E. Manis [**8**]. Especially if $A$ is a Prüfer domain or almost Dedekind domain, we determine to what extent these properties hold for $\hat{A}$.

In this paper all rings and groups are assumed to be commutative. At first, we recall some results of Manis [**8**] and M. Griffin [**4**]. A *valuation* on a ring $R$ is a map $w$ from $R$ onto a totally ordered group $G$ and a symbol $\infty$, such that for all $a, b \in R$,

(i) $w(ab) = w(a) + w(b)$,

(ii) $w(a + b) \geqslant \min\{w(a), w(b)\}$.

Note that in (ii) equality holds if $w(a) \neq w(b)$. The *valuation ring* $R_w$ of $w$ is defined to be the subring of $R$ of all elements with nonnegative value. The set $M(w)$ of all elements of $R$ with positive value is a prime ideal of $R_w$ and $(R_w, M(w))$ is said to be a *valuation pair* associated with $w$. An ideal $Q$ of a valuation ring $R_w$ is *w-closed* if $x \in Q$, $y \in R_w$, and $w(x) \leqslant w(y)$ imply $y \in Q$. M. Griffin [**4**] defines the *large quotient ring* of a ring $R$ with respect to a multiplicative system $S$ of $R$ as $R_{[S]} = \{x \in K: xs \in R \text{ for some } s \in S\}$, where $K$ is the total quotient ring of $R$. If $P$ is a prime ideal of $R$ with

Received by the editors May 12, 1976 and, in revised form, April 25, 1977.
*AMS (MOS) subject classifications* (1970). Primary 13A15, 13J10.

© American Mathematical Society 1977

$S = R - P$, $R_{[P]}$ will be used for $R_{[S]}$. Also $[P]R_{[P]}$ denotes the prime ideal of $R_{[P]}$ defined by $\{x \in K: xs \in P \text{ for some } s \in R - P\}$. Then a *Prüfer ring* is a ring in which every finitely generated regular ideal is invertible. By [**4**, Theorem 13], $R$ is a Prüfer ring if and only if for every maximal regular ideal $Q$ of $R$, $(R_{[Q]}, [Q]R_{[Q]})$ is a valuation pair associated with some valuation on $K$.

M. D. Larsen [**7**] introduced the notion of an *N-ring* which generalizes almost Dedekind domain, where an almost Dedekind domain is an integral domain $A$ such that for all prime ideals $P$ of $A$, $A_P$ is a discrete rank one valuation ring. He defines an $N$-ring to be a ring $R$ in which for all maximal regular prime ideals $Q$ of $R$, $(R_{[Q]}, [Q]R_{[Q]})$ is a valuation pair associated with a valuation $w$ such that $G_w$ is isomorphic to the group $Z$ of integers.

The author is grateful to the referee, especially for his suggestions concerning $h$-local Prüfer domains.

**2.** We begin this section with some results about the topology induced by a Manis valuation. Let $(R_w, M(w))$ be a valuation pair and let $K$ be the total quotient ring of $R_w$. Then the sets

$$U_{w,\alpha} = \{x \in K: w(x) > \alpha\}, \qquad \alpha \in G_w^+,$$

form a base of the neighbourhoods of zero in $K$ for some ring topology $\mathcal{T}_w$ in $K$. For, $U_{w,\alpha}$ is an additive subgroup of $K$, $U_{w,\alpha} \cdot U_{w,\alpha} \subseteq U_{w,\alpha}$, and for any $x_0 \in K$, $\alpha \in G_w^+$, we have $x_0 U_{w,\beta} \subseteq U_{w,\alpha}$, where $\beta \geqslant \alpha - w(x_0)$ if $x_0 \notin w^{-1}(\infty)$ and $\beta \in G_w^+$ for $x_0 \in w^{-1}(\infty)$. Now, if we define on the set $G_w' = G_w \cup \{\infty\}$ a topology by setting $\overline{X} = X \cup \{\infty\}$ for every $X \subseteq G_w'$, $X \neq \varnothing$, $\overline{\varnothing} = \varnothing$, then $G_w'$ is a complete uniform space and $w$: $(K, \mathcal{T}_w) \to G_w'$ is continuous.

LEMMA 1. *Let $R_w$ be a valuation ring with $J$ an ideal of $R_w$. If $J$ is $w$-closed and $J \neq w^{-1}(\infty)$, it is clopen in $R_w$. If $J$ is closed, then $w^{-1}(\infty) \subseteq J$.*

PROOF. Let $J$ be $w$-closed Since $J \neq w^{-1}(\infty)$, there exists $a \in J - w^{-1}(\infty)$. Then for every $x \in J$ and $y \in U_{w,w(a)}$ we have either $w(x + y) \geqslant w(x)$ or $w(x + y) \geqslant w(a)$, hence $x + U_{w,w(a)} \subseteq J$ and $J$ is open in $R_w$. Since $J$ is a subgroup of $R_w$, $J$ is clopen. Now let $J$ be closed in $R_w$ and let $x \in w^{-1}(\infty)$. Then $x \in \overline{\{0\}}$, the closure of $\{0\}$ in $R_w$, and $\overline{\{0\}} \subseteq J$.

LEMMA 2. *A ring $K$ contains no zero divisors if there exists a nontrivial valuation on $K$ such that the associated topology is Hausdorff.*

PROOF. Suppose that there exists a nontrivial valuation $w$ on $K$ such that $(K, \mathcal{T}_w)$ is Hausdorff. Then $w^{-1}(\infty) \subseteq \overline{\{0\}} = \{0\}$. If there is a zero divisor in $K$, then there exists $x \in w^{-1}(\infty)$, $x \neq 0$, which cannot happen.

Further, suppose that $A$ is an integral domain with the quotient field $K$ and let $\Omega$ be the family of nontrivial valuations on $K$ which are nonnegative on $A$ and put

$$\mathcal{T} = \sup\{\mathcal{T}_w: w \in \Omega\}.$$

If $(\hat{K}_w, \hat{\mathcal{T}}_w)$ and $(\hat{K}, \hat{\mathcal{T}})$ are the completions of $(K, \mathcal{T}_w)$ and $(K, \mathcal{T})$, respectively, we denote by $\hat{w}$, $\tilde{w}$ the continuous extension of $w$ on $\hat{K}_w$ and $\hat{K}$, respectively. It is well known that $\hat{w}$ is a valuation on the field $\hat{K}_w$ and $\hat{\mathcal{T}}_w = \mathcal{T}_{\hat{w}}$.

LEMMA 3. *$\tilde{w}$ is a Manis valuation on $\hat{K}$ for any $w \in \Omega$.*

PROOF. We obtain the result from the fact that $K$ is a dense subset in the Hausdorff space $\hat{K}$ and $\tilde{w}$ is continuous.

Now, since $\tilde{w}$ is a Manis valuation on $\hat{K}$, it defines a topology $\mathcal{T}_{\tilde{w}}$ on $\hat{K}$. Then we can prove the following:

PROPOSITION 4. $\hat{\mathcal{T}} = \sup\{\mathcal{T}_{\tilde{w}} : w \in \Omega\}$.

PROOF. We consider the following diagram

$$\begin{tikzcd}
K \arrow[r, "f_w"] \arrow[d, "i"'] & K_w = (K, \mathcal{T}_w) \arrow[d, "i_w"] \\
\hat{K} \arrow[r, "\hat{f}_w"] \arrow[dr, "\tilde{w}"'] & \hat{K}_w \arrow[d, "\hat{w}"] \\
 & G_w
\end{tikzcd}$$

where $w \in \Omega$, $\hat{f}_w$ is the continuous extension of $f_w = \mathrm{id}_K$ and $i, i_w$ are the canonical injections. We have $\tilde{w} \cdot i = w$, $\hat{f}_w \cdot i = i_w \cdot f_w$, $w = \hat{w} \cdot i_w \cdot f_w$; thus $\tilde{w} \cdot i = \hat{w} \cdot \hat{f}_w \cdot i$. Since $i(K)$ is a dense subset in the Hausdorff space $\hat{K}$, we obtain $\tilde{w} = \hat{w} \cdot \hat{f}_w$. Now, since $U_{\tilde{w},\alpha} = \hat{f}_w^{-1}(U_{\hat{w},\alpha})$ for every $\alpha \in G_w^+$, $w \in \Omega$, $U_{\hat{w},\alpha}$ is open in $\hat{K}_w$; it follows that $U_{\tilde{w},\alpha}$ is open in $\hat{\mathcal{T}}$; thus $\hat{\mathcal{T}} \geqslant \mathcal{T}_{\tilde{w}}$ for every $w \in \Omega$. Suppose that $\mathcal{G}$ is a topology on $\hat{K}$ such that $\mathcal{G} \geqslant \mathcal{T}_{\tilde{w}}$ for every $w \in \Omega$. Then from the fact that $\tilde{w} = \hat{w} \cdot \hat{f}_w$, we obtain that $\hat{f}_w$ is continuous in $(\hat{K}, \mathcal{G})$ for every $w \in \Omega$. Thus, using the fact $\mathcal{T} = \sup\{\mathcal{T}_w : w \in \Omega\}$, we obtain by [**2**, §3, Proposition 18], that the topology $\hat{\mathcal{T}}$ is the supremum of topologies on $\hat{K}$ for which $\hat{f}_w$, $w \in \Omega$, are continuous. Hence, $\hat{\mathcal{T}} \leqslant \mathcal{G}$ and $\hat{\mathcal{T}} = \sup\{\mathcal{T}_{\tilde{w}} : w \in \Omega\}$.

Henceforth, we assume that $K$ is a subring of $\hat{K}$.

PROPOSITION 5. *Let $w \in \Omega$ and let $(R_{\tilde{w}}, M(\tilde{w}))$ be the valuation pair associated with $\tilde{w}$. Let $R_w$ be a valuation ring with the maximal ideal $M(w)$ of a valuation $w$ in a field $K$. Then*

$$R_{\tilde{w}} = \overline{R_w}, \qquad M(\tilde{w}) = \overline{M(w)},$$

*where $\overline{X}$ is the closure of $X$ in $(\hat{K}, \hat{\mathcal{T}})$.*

PROOF. Let $x \in \hat{K} - \overline{R_w}$. Then by Proposition 4, there exist $w_1, \ldots, w_n \in \Omega$, $\alpha_i \in G_{w_i}^+$, $i = 1, \ldots, n$, such that

$$\left(x + \bigcap_{i=1}^{n} U_{\tilde{w}_i, \alpha_i}\right) \cap R_w = \varnothing.$$

There is no loss of generality in assuming that $w \in \{w_1, \ldots, w_n\}$, say $w = w_1$. Moreover, since $K$ is a dense subset in $\hat{K}$, there exists $z \in K$ such that

$$\tilde{w}_i(z - x) > \alpha_i, \qquad i = 1, \ldots, n.$$

Hence, $z \in K - R_w$ and $w(z) = w_1(z) < 0$. Suppose that $w_1(z) \neq \tilde{w}_1(x)$. Then

$$\alpha_1 < \tilde{w}_1(z - x) = \min\{\tilde{w}_1(x), w_1(z)\} < 0,$$

a contradiction with $\alpha_1 \geqslant 0$. Thus, $\tilde{w}(x) = w(z) < 0$ and $x \notin R_{\tilde{w}}$. Therefore, $R_{\tilde{w}} \subseteq \overline{R_w}$. On the other hand, $R_{\tilde{w}}$ is closed in $\hat{K}$ and we have $\overline{R_w} \subseteq R_{\tilde{w}}$. Thus, $R_{\tilde{w}} = \overline{R_w}$. The rest can be proved analogously.

For future investigation we need to solve the following problem. If $w$ is a valuation on the quotient field $K$ of $A$ such that $R_w = A_{P(w)}$, where $P(w) = M(w) \cap A$, when $R_{\hat{w}} = (\hat{A}_w)_{P(\hat{w})}$ holds, where $\hat{A}_w$ is the closure of $A$ in $\hat{K}_w$ and $P(\hat{w}) = M(\hat{w}) \cap \hat{A}_w$. The following example shows that it does not hold in general.

EXAMPLE. Let $Q$ be the field of rational numbers and let $X$ be an indeterminate over $Q$. In the field $Q(X)$ we define a valuation $w$ in the following way.

$$w\left(\sum_{i=0}^{n} a_i X^i\right) = k \Leftrightarrow a_0 = \cdots = a_{k-1} = 0, \qquad a_k \neq 0;$$

$$w(f/g) = w(f) - w(g), \qquad f, g \in Q[X], g \neq 0, w(0) = \infty.$$

Then $R_w$ is a discrete rank one valuation ring and $R_w = Z[X]_{M(w) \cap Z[X]}$. Now in the quotient field $Q((X))$ of the ring $Q[[X]]$ of formal power series over $Q$, we define a valuation $\hat{w}$ in the following way.

$$\hat{w}\left(\sum_{n=0}^{\infty} a_n X^n\right) = k \Leftrightarrow a_0 = \cdots = a_{k-1} = 0, \qquad a_k \neq 0, a_n \in Q;$$

$$\hat{w}(f/g) = \hat{w}(f) - \hat{w}(g), \qquad f, g \in Q[[X]], g \neq 0, \hat{w}(0) = \infty.$$

Then $(Q((X)), \mathcal{T}_{\hat{w}})$ is the completion of $(Q(X), \mathcal{T}_w)$, $\hat{w}$ is the unique continuous extension of $w$ on $Q((X))$ and $Z[[X]]$ is the closure of $Z[X]$ in $Q((X))$. To show that $R_{\hat{w}} \neq Z[[X]]_{M(\hat{w}) \cap Z[[X]]}$, we need a simple lemma.

LEMMA. *Let $n$ be a natural number, $z_i \in Z^+$, $i = 0, \ldots, n-1$, be such that $0 \leqslant z_i < p_{n-i}$, where $p_j$ is the $j$th prime number and let $z_0(1/p_n) + z_1(1/p_{n-1}) + \cdots + z_{n-1}(1/2) \in Z$. Then $z_0 = z_1 = \cdots = z_{n-1} = 0$.*

PROOF. The proof is by induction on $n$. The lemma holds for $n = 1$. We assume that the lemma holds for $n$. Let $z_i \in Z$, $i = 0, \ldots, n$, be such that $0 \leqslant z_i < p_{n+1-i}$, and let $z_0(1/p_{n+1}) + \cdots + z_n(1/2) \in Z$. Let $z_i'$, $i =$

$0, \ldots, n-1$, be such that

$$z_{i+1}p_{n+1} = z_i' \pmod{p_{n-i}}, \qquad 0 \leqslant z_i' < p_{n-i}.$$

Then $z_0'(1/p_n) + \cdots + z_{n-1}'(1/2) \in Z$, and by induction we have $z_0' = \cdots = z_{n-1}' = 0$. Hence, $z_1 p_{n+1}/p_n \in Z, \ldots, z_n p_{n+1}/2 \in Z$, and we obtain $z_0 = z_1 = \cdots = z_n = 0$.

Now, $f = 1 + (1/2)X + \cdots + (1/p_n)X^n + \cdots \in R_{\hat{w}}$. We suppose that there exist $u = \Sigma a_n X^n$, $v = \Sigma b_n X^n \in Z[[X]]$ such that $a_0 \neq 0$ and $v = f \cdot u$. Then for any natural $n$ we have

$$b_n = a_n + (1/2)a_{n-1} + \cdots + (1/p_n)a_0.$$

Let $a_{n-i} = z_{n-i} (\text{mod } p_i)$, $i = 1, \ldots, n$, $0 \leqslant z_{n-i} < p_i$. Then $z_0(1/p_n) + \cdots + z_{n-1}(1/2) \in Z$, and by the lemma we have $z_0 = \cdots = z_{n-1} = 0$. Thus, $p_n$ divides $a_0$ for every $n$, a contradiction.

The partial solution of the problem gives the following two propositions.

PROPOSITION 6. *Let $w$ be a discrete rank one valuation on the quotient field $K$ of $A$ such that $R_w = A_{P(w)}$ and let $P(w)$ be a maximal ideal of $A$. Then $R_{\hat{w}} = \hat{A}_w$.*

PROOF. It is well known that in this case $\mathfrak{T}_w$ is the $M(w)$-adic topology. Then $\hat{A}_w$ is the completion of $A$ in $P(w) = M(w) \cap A$-adic topology and, since $P(w)$ is maximal in $A$, $\hat{A}_w$ is local with the unique maximal ideal $P(\hat{w}) = M(\hat{w}) \cap \hat{A}_w$. Hence, $R_w = A_{P(w)} \subseteq (\hat{A}_w)_{P(\hat{w})} = \hat{A}_w$ and $R_{\hat{w}} = \overline{R_w} \subseteq \hat{A}_w \subseteq R_{\hat{w}}$, where the vinculum denotes the closure in $\hat{K}_w$. Therefore, $R_{\hat{w}} = \hat{A}_w$.

Using the example it should be observed that "$P(w)$ be maximal" cannot be removed from Proposition 6.

The following proposition follows immediately from [**10**, Theorem 2.1].

PROPOSITION 7. *$R_{\hat{w}} = (\hat{A}_w)_{P(\hat{w})}$ if and only if there exists an order epimorphism $f$ from a group of divisibility $G(\hat{A}_w)$ of $\hat{A}_w$ onto $G_w$ such that $f \cdot v = \hat{w}$, where $v$ is the canonical map from $\hat{K}_w^*$ onto $G(\hat{A}_w)$.*

Further, if $R_{\hat{w}} = (\hat{A}_w)_{P(\hat{w})}$ for a valuation $w$, it is natural to ask when $R_{\tilde{w}} = \hat{A}_{[P(\tilde{w})]}$ for $P(\tilde{w}) = M(\tilde{w}) \cap \hat{A}$. To solve this problem, we prove two lemmas.

At first, on the family $\Omega$ we may define an equivalence relation

$$w \equiv w' \quad \text{if and only if } w, w' \text{ are dependent.}$$

Let $\Omega_0$ be a family of representatives of the equivalence classes.

LEMMA 8. *Let for every $w \in \Omega_0$, $x_w$ be an element from $\hat{K}_w$. Then there exists $x \in \hat{K}$ such that $\hat{f}_w(x) = x_w$ for every $w \in \Omega_0$. Furthermore, if $x_w \neq 0$ for every $w \in \Omega_0$, $x$ is regular.*

PROOF. We set $\Delta = \Pi\{i_w\colon w \in \Omega_0\}$, where $i_w\colon K \to \hat{K}_w$ is the canonical map. Then using the approximation theorem for independent valuations, we obtain that

$$\Delta\colon K \to \Pi\{\hat{K}_w\colon w \in \Omega_0\}$$

is a completion of the topological field $(K, \mathfrak{T})$, and the canonical projection map $\mathrm{pr}_w$ of $\Pi\{\hat{K}_w\colon w \in \Omega_0\}$ onto $\hat{K}_w$ is the unique continuous homomorphism such that

$$i_w \cdot f_w = \mathrm{pr}_w \cdot \Delta, \qquad w \in \Omega_0.$$

Now, since $i\colon K \to \hat{K}$ is the completion of $K$, there exists an isomorphism $\varphi$ such that $\varphi \cdot \Delta = i$. Then $(\hat{f}_w \cdot \varphi) \cdot \Delta = \hat{f}_w \cdot i = i_w \cdot f_w$, hence $\mathrm{pr}_w = \hat{f}_w \cdot \varphi$ for every $w \in \Omega_0$. Let $(x_w) \in \Pi\{\hat{K}_w\colon w \in \Omega_0\}$. We set $x = \varphi((x_w)) \in \hat{K}$. Then $\hat{f}_w(x) = \mathrm{pr}_w((x_w)) = x_w$ and if $x_w \neq 0$ for every $w \in \Omega_0$, $(x_w)$ is regular in $\Pi\{\hat{K}_w\colon w \in \Omega_0\}$. Thus, $x$ is regular in $\hat{K}$.

For the second lemma we need some notation from [5]. Let $w, w' \in \Omega$. If $R_w \subseteq R_{w'}$ we say that $w'$ is coarser than $w$ and write $w' \leqslant w$. If $w' \leqslant w$ then $G_{w'} \cong G_w/H$, where $H$ is an isolated subgroup of $G_w$. Since the valuations coarser than $w$ are totally ordered there will be a finest valuation $w'' = w' \wedge w$ coarser than both $w, w'$. Let $H$ be a corresponding subgroup of $G_w$. Then we denote by ${}_{(w,w')}\bar{\alpha}$ the element $\psi(\alpha)$, where $\psi\colon G_w \to G_{w''}$ is the canonical map and $\alpha \in G_w$. Now, let $w_1, \ldots, w_n \in \Omega$. Let $(\alpha_1, \ldots, \alpha_n) \in G_{w_1} \times \cdots \times G_{w_n}$, $b_i \in K$, $1 \leqslant i \leqslant n$. We define $((\alpha_1, \ldots, \alpha_n), (b_1, \ldots, b_n))$ to be *concordant* when the following conditions hold:

(1) if ${}_{(i,j)}\bar{\alpha}_i = {}_{(j,i)}\bar{\alpha}_j$ then ${}_{(i,j)}\bar{\alpha}_i \leqslant \overline{{}_{(j,i)}w_j(b_i - b_j)}$;

(2) if ${}_{(i,j)}\bar{\alpha}_i > {}_{(j,i)}\bar{\alpha}_j$ then ${}_{(j,i)}\bar{\alpha}_j = \overline{{}_{(j,i)}w_j(b_i - b_j)}$,

where ${}_{(i,j)}\bar{\alpha} = {}_{(w_i,w_j)}\bar{\alpha}$. Then we say that $\Omega$ satisfies the *weak reinforced approximation theorem* (W.R.A.T.) for $A$ if, for any finite number $w_i \in \Omega$, $i = 1, \ldots, n$, of valuations with $(\alpha_1, \ldots, \alpha_n) \in G_{w_1}^+ \times \cdots \times G_{w_n}^+$ and $(b_1, \ldots, b_n) \in A^n$ such that $((\alpha_1, \ldots, \alpha_n), (b_1, \ldots, b_n))$ is concordant, there exists $a \in A$ such that

$$w_i(a - b_i) = \alpha_i, \qquad i = 1, \ldots, n.$$

By [5, Proposition 24], for every Prüfer domain $A$, $\Omega$ satisfies the W.R.A.T.

LEMMA 9. *Let $\Omega$ satisfy the W.R.A.T. for $A$. Then*

$$\hat{A} = \bigcap\{\hat{f}_w^{-1}(\hat{A}_w)\colon w \in \Omega_0\}.$$

PROOF. At first, since $\hat{f}_w$ is the continuous extension of $f_w = \mathrm{id}_K$ for every $w$, we have $\hat{f}_w(\hat{A}) \subseteq \overline{f_w(A)} = \hat{A}_w$. Conversely, let $x \in \hat{f}_w^{-1}(\hat{A}_w)$ for every $w \in \Omega_0$. Let $w_i \in \Omega_0$, $i = 1, \ldots, n$, $\alpha_i \in G_{w_i}^+$. Then since $A$ is a dense subset in $\hat{A}_{w_i}$, for every $i$, $1 \leqslant i \leqslant n$, we may find an element $y_i$ such that

$$y_i \in A \cap \left(\hat{f}_{w_i}(x) + U_{\tilde{w}_i,\alpha_i}\right).$$

Now, since $w_i$ are pairwise independent, $((\alpha_1, \ldots, \alpha_n), (y_1, \ldots, y_n))$ is concordant. Thus, there exists $a \in A$ such that

$$w_i(a - y_i) > \alpha_i, \qquad i = 1, \ldots, n.$$

Then using the identity $\tilde{w}_i = \hat{w}_i \cdot \hat{f}_{w_i}$ from the proof of Proposition 4, we have

$$\tilde{w}_i(a - x) = \hat{w}_i \cdot \hat{f}_{w_i}(a - x) = \hat{w}_i\big(a - \hat{f}_{w_i}(x)\big)$$
$$= \hat{w}_i\big(a - y_i + y_i - \hat{f}_{w_i}(x)\big) > \alpha_i, \qquad 1 \leqslant i \leqslant n.$$

Therefore, $A$ is a dense subset in $\bigcap\{\hat{f}_w^{-1}(\hat{A}_w)\colon w \in \Omega_0\}$.

PROPOSITION 10. *Let $w \in \Omega$ be such that $R_{\tilde{w}} = \hat{A}_{[P(\tilde{w})]}$. Then $R_{\tilde{w}} = (\hat{A}_w)_{P(\hat{w})}$. Conversely, if $\Omega$ satisfies the W.R.A.T. for $A$ and if $R_{\tilde{w}} = (\hat{A}_w)_{P(\hat{w})}$, then*

$$R_{\tilde{w}} = \hat{A}_{[P(\tilde{w})]}, \qquad M(\tilde{w}) = [P(\tilde{w})]\hat{A}_{[P(\tilde{w})]}.$$

PROOF. Let $R_{\tilde{w}} = \hat{A}_{[P(\tilde{w})]}$ and let $x \in R_{\tilde{w}}$. Then for $y \in \hat{f}_w^{-1}(x)$ there exists $z \in \hat{A} - P(\tilde{w})$ such that $y \cdot z \in \hat{A}$. Hence, $x \cdot \hat{f}_w(z) \in \hat{A}_w$ and $\hat{f}_w(z) \in \hat{A}_w - P(\hat{w})$. Thus, $x \in (\hat{A}_w)_{P(\hat{w})}$. The converse inclusion is trivial. Conversely, let $R_{\tilde{w}} = (\hat{A}_w)_{P(\hat{w})}$. Let $\Omega_0$ be the family of representatives of the equivalence classes of $\equiv$ such that $w \in \Omega_0$. Let $x \in R_{\tilde{w}}$. Then there exists $y_w \in \hat{A}_w - P(\hat{w})$ such that $y_w \cdot \hat{f}_w(x) \in \hat{A}_w$. Further, since $\hat{K}_{w'}$ is the quotient field of a domain $\hat{A}_{w'}$ for every $w' \in \Omega_0$, there exist $y_{w'} \in \hat{A}_{w'}$, $w' \in \Omega_0$, $w' \neq w$, such that $y_{w'} \cdot \hat{f}_{w'}(x) \in \hat{A}_{w'}$. Now, by Lemma 8, there exists $y \in \hat{K}$ such that $\hat{f}_{w'}(y) = y_{w'}$, $w' \in \Omega_0$. Then by Lemma 9, $y \in \hat{A}$, and using the identity $\tilde{w} = \hat{w} \cdot \hat{f}_w$, $y \in \hat{A} - P(\tilde{w})$. Further,

$$\hat{f}_{w'}(y \cdot x) = y_{w'} \cdot \hat{f}_{w'}(x) \in \hat{A}_{w'}, \qquad w' \in \Omega_0,$$

hence $y \cdot x \in \hat{A}$. Therefore, $x \in \hat{A}_{[\hat{A} - P(\tilde{w})]}$. The converse inclusion is trivial. Further, it is clear that $[P(\tilde{w})]\hat{A}_{[P(\tilde{w})]} \subseteq M(\tilde{w})$. Let $y \in M(\tilde{w}) \subseteq \hat{A}_{[P(\tilde{w})]}$. Then there exists $z \in \hat{A} - P(\tilde{w})$ such that $y \cdot z \in \hat{A}$ and it is easy to see that $y \cdot z \in P(\tilde{w})$. Thus, $[P(\tilde{w})]\hat{A}_{[P(\tilde{w})]} = M(\tilde{w})$.

LEMMA 11. *Let $\Omega$ satisfy the W.R.A.T. for $A$. Then $\hat{A} = \bigcap\{R_{\tilde{w}}\colon w \in \Omega_0\}$ if and only if $A$ is a dense subset in $(R_w, \mathcal{T}_w|R_w)$ for every $w \in \Omega_0$.*

PROOF. If $A$ is a dense subset in $R_w$, then $\hat{A}_w = R_{\hat{w}}$ and the rest follows by Lemma 9. Conversely, let $\hat{A} = \bigcap\{R_{\tilde{w}}\colon w \in \Omega_0\}$. If $A$ is not a dense subset in $R_w$ for some $w \in \Omega_0$, then $\hat{A}_w \neq R_{\hat{w}}$, and there exists $x_w \in R_{\tilde{w}}$ such that $\hat{f}_w(x_w) \notin \hat{A}_w$. Further, let for every $w' \in \Omega_0$, $w' \neq w$, $x_{w'} \in R_{\tilde{w}'}$. By Lemma 8, there exists $x \in \hat{K}$ such that $\hat{f}_{w'}(x) = x_{w'}$ for every $w' \in \Omega_0$. Thus, $x \in \hat{f}_{w'}^{-1}(R_{\tilde{w}'}) = R_{\tilde{w}'}$, $w' \in \Omega_0$. Hence, $x \in \hat{A}$ and $x_w \in \hat{A}_w$, a contradiction.

THEOREM 12. *Let $A$ be a Prüfer domain such that there exists a family $\Omega_0$ of representatives of the equivalence classes of $\equiv$ such that $A$ is a dense subset in $(R_w, \mathcal{T}_w|R_w)$ for every $w \in \Omega_0$. Then $\hat{A}$ is a Prüfer ring.*

PROOF. By Lemma 11, $\hat{A} = \bigcap\{R_{\tilde{w}}\colon w \in \Omega_0\}$. Let $a, b \in \hat{A}$ be such that the ideal $(a, b)$ in $\hat{A}$ is regular. Since $R_{\tilde{w}}$ is a valuation domain, for every $w \in \Omega_0$ there exists $x_w \in R_{\tilde{w}}$ such that $(\hat{f}_w(a), \hat{f}_w(b))R_{\tilde{w}} = x_w R_{\tilde{w}}$. By Lemma 8, there exists $x \in \hat{K}$ such that $\hat{f}_w(x) = x_w$ for every $w \in \Omega_0$. Hence $x \in R_{\tilde{w}}$ for every $w \in \Omega_0$ and $x \in \hat{A}$. Then using the identity $\mathrm{pr}_w = \hat{f}_w \cdot \varphi$ from the proof of Lemma 8, we obtain $(a, b) = x\hat{A}$, where $x$ is regular. Thus $(a, b)$ is invertible and, by the definition, $\hat{A}$ is a Prüfer ring.

It should be observed that there is a reasonably large class of Prüfer domains which satisfy the conditions of Theorem 12. In fact, every $h$-local Prüfer domain is of such type. Recall that an integral domain $A$ is said to be $h$-local if every nonzero ideal of $A$ is contained in only a finite number of maximal ideals, and if every nonzero prime ideal of $A$ is contained in only one maximal ideal. Then by [**9**, Theorem 22], $A$ is $h$-local if and only if $\hat{A} \cong \Pi\{\hat{A}_M\colon M \in \operatorname{mspec} A\}$, where mspec $A$ is the set of maximal ideals of $A$, $\hat{A}$ is the completion of $A$ in the $A$-topology, and $\hat{A}_M$ is the completion of $A_M$ in the $A_M$-topology, where for any ring $R$ the $R$-topology is the topology on $R$ with the ideals $rR$, $r \in R^*$, being a subbase for the open neighbourhoods of 0 in $R$.

The following lemma holds.

LEMMA 13. *Let $A$ be a $h$-local Prüfer domain. Then the topology* $\sup\{\mathfrak{T}_w\colon w \in \Omega\}$ *on $A$ is the same as the $A$-topology.*

PROOF. Let $w_i \in \Omega$, $\alpha_i \in G_{w_i}^+$. Then there exist $a_i \in A$, $i = 1, \ldots, n$, such that $w_i(a_i) > \alpha_i$. Then

$$a_1A \cap \cdots \cap a_nA \subseteq \bigcap_{i=1}^{n} U_{w_i,\alpha_i}.$$

Conversely, let $a \in A^*$. Then since $A$ is $h$-local, there exist only a finite number of maximal ideals $M_1, \ldots, M_n \in \operatorname{mspec} A$ such that $a \in M_i$. Let $w_i \in \Omega$, $i = 1, \ldots, n$, be such that $M_i = P(w_i)$, and we set $\alpha_i = w_i(a)$. Let $x \in A \cap \bigcap_{i=1}^{n} U_{w_i,\alpha_i}$. If $x \notin aA$, then there exists $w \in \Omega$ such that $w(xa^{-1}) < 0$. Since $w \neq w_i$, we have $a \notin P(w)$ and $w(xa^{-1}) = w(x) \geqslant 0$, a contradiction. Thus, $\bigcap_{i=1}^{n} U_{w_i,\alpha_i} \cap A \subseteq aA$ and $\sup\{\mathfrak{T}_w\}$ on $A$ is the $A$-topology.

Since for every $h$-local Prüfer domain $A$, the family $\Omega_0 = \{w \in \Omega\colon P(w) \in \operatorname{mspec} A\}$ is a set of representatives of the equivalence classes of $\equiv$, $A$ satisfies the conditions of Theorem 12 and $\hat{A}$ is a Prüfer ring.

Further, we say that a ring $A$ with the total quotient ring $K$ is a *$\mathfrak{F}$-Prüfer ring*, where $\mathfrak{F}$ is a ring topology on $K$ if, for every maximal regular ideal $\mathfrak{m}$ of $A$, $(A_{[\mathfrak{m}]}, [\mathfrak{m}]A_{[\mathfrak{m}]})$ is a valuation pair associated with a valuation $w$ on $K$ such that $w$ is continuous in $\mathfrak{F}$.

To generalize the definition of $w$-closed ideals, we say that an ideal $Q$ of $A$ is *$\tilde{w}/\hat{A}$-closed* for some $w \in \Omega$ if $x \in Q$, $y \in A$, and $\tilde{w}(x) \leqslant \tilde{w}(y)$ imply $y \in Q$.

We are now ready to prove the main theorem.

THEOREM 14. *Let $A$ be a Prüfer domain. Then the following conditions are equivalent.*

(1) *$\hat{A}$ is a $\hat{\mathfrak{T}}$-Prüfer ring.*

(2) *For every maximal regular ideal $\mathfrak{m}$ of $\hat{A}$ there exists $w \in \Omega$ such that $\mathfrak{m}$ is $\tilde{w}/\hat{A}$-closed and $R_{\hat{w}} = (\hat{A}_w)_{P(\hat{w})}$ for every $w \in \Omega$.*

(3) *Every maximal regular ideal of $\hat{A}$ is open in $\hat{A}$ and $R_{\hat{w}} = (\hat{A}_w)_{P(\hat{w})}$ for every $w \in \Omega$.*

PROOF. (1) $\Rightarrow$ (2). Let $\mathfrak{m}$ be a maximal regular ideal of $\hat{A}$. Then $(\hat{A}_{[\mathfrak{m}]}, [\mathfrak{m}]\hat{A}_{[\mathfrak{m}]})$ is a valuation pair associated with a valuation $v$ continuous in $\hat{\mathfrak{T}}$. Let $w = v|K$. It is easy to see that $w$ is a valuation on $K$. If $w$ is the trivial one, it follows since $w$ is continuous in $\mathfrak{T}$, that $\mathfrak{T}$ is discrete. Thus, $\hat{K} = K$, $\hat{A} = A$, $\mathfrak{m} = M(w) \cap A = (0)$, a contradiction with $\mathfrak{m}$ regular. Hence $w$ is nontrivial, $w \in \Omega$, $\tilde{w}$ and $v$ are the continuous extensions of $w$ on $\hat{K}$. Since $\hat{K}$ is Hausdorff, we obtain $\tilde{w} = v$. Hence $R_{\tilde{w}} = \hat{A}_{[\mathfrak{m}]}$, $M(\tilde{w}) = [\mathfrak{m}]\hat{A}_{[\mathfrak{m}]}$, and it is easy to see that $\hat{A}_{[\mathfrak{m}]} = \hat{A}_{[P(\tilde{w})]}$, $P(\tilde{w}) = \mathfrak{m}$. Thus, $\mathfrak{m}$ is $\tilde{w}/\hat{A}$-closed and the rest follows by Proposition 10.

(2) $\Rightarrow$ (3). Let $\mathfrak{m}$ be a maximal regular ideal of $\hat{A}$ and let $x \in \mathfrak{m}$ be regular. Then for $z \in U_{\tilde{w},\tilde{w}(x)} \cap \hat{A}$ we have $\tilde{w}(y + z) \geqslant \tilde{w}(z) \geqslant \tilde{w}(x)$ if $\tilde{w}(y) \geqslant \tilde{w}(z)$ and $\tilde{w}(y + z) = \tilde{w}(y)$ if $\tilde{w}(y) < \tilde{w}(z)$. Hence $y + z \in \mathfrak{m}$, and $\mathfrak{m}$ is open in $\hat{A}$.

(3) $\Rightarrow$ (1). Let $\mathfrak{m}$ be a maximal regular ideal of $\hat{A}$ and $P = A \cap \mathfrak{m}$. Since $A$ is a dense subset in $\hat{A}$, and $\mathfrak{m}$ is open in $\hat{A}$, it follows that $\mathfrak{m} = \overline{\mathfrak{m} \cap A} = \bar{P}$, where the vinculum denotes the closure in $\hat{A}$. Since $\mathfrak{m}$ is regular, we have $P \neq (0)$. Let $w \in \Omega$ be such that $R_w = A_P$. By Proposition 10, $R_{\tilde{w}} = \hat{A}_{[P(\tilde{w})]}$. It is clear that $\bar{P} \subseteq P(\tilde{w})$. Suppose that there exists $x \in P(\tilde{w}) - \bar{P}$; then for some $w_1, \ldots, w_n \in \Omega$, $\alpha_i \in G_{w_i}^+$, $i = 1, \ldots, n$, we have

$$\left(x + \bigcap_{i=1}^{n} U_{\tilde{w}_i,\alpha_i}\right) \cap P = \varnothing.$$

On the other hand, there exists $a \in A$ such that

$$\tilde{w}_i(a - x) > \alpha_i, \qquad i = 1, \ldots, n.$$

Again, there is no loss of generality in assuming that $w \in \{w_1, \ldots, w_n\}$, say $w = w_1$. Then since $P = P(w_1)$, it follows that

$$\alpha_1 < \tilde{w}_1(a - x) = \min\{w_1(a), \tilde{w}_1(x)\} = 0,$$

a contradiction. Thus, $P(\tilde{w}) = \bar{P} = \mathfrak{m}$, and $(\hat{A}_{[\mathfrak{m}]}, [\mathfrak{m}]\hat{A}_{[\mathfrak{m}]})$ is a valuation pair associated with the valuation $\tilde{w}$ continuous in $\hat{\mathfrak{T}}$. Therefore, $\hat{A}$ is a $\hat{\mathfrak{T}}$-Prüfer ring.

We have not been able to show that there exists a Prüfer domain $A$ such that $\hat{A}$ is not a $\hat{\mathfrak{T}}$-Prüfer ring, and we do not know if there is a Prüfer domain $A$ such that $\hat{A}$ is a Prüfer ring but $\hat{A}$ is not a $\hat{\mathfrak{T}}$-Prüfer ring.

Finally, we say that a ring $A$ with the total quotient ring $K$ is a *$\mathfrak{F}$-N-ring* for some ring topology $\mathfrak{F}$ on $K$, if for every maximal regular ideal $\mathfrak{m}$ of $A$, $(A_{[\mathfrak{m}]}, [\mathfrak{m}]A_{[\mathfrak{m}]})$ is a valuation pair associated with a valuation $w$ on $K$ continuous in $\mathfrak{F}$ and such that $G_w \cong Z$.

THEOREM 15. *Let $A$ be an almost Dedekind domain. Then the following conditions are equivalent.*

(1) *$\hat{A}$ is a $\hat{\mathfrak{T}}$-N-ring.*

(2) *For every maximal regular ideal $\mathfrak{m}$ of $\hat{A}$ there exists $w \in \Omega$ such that $\mathfrak{m}$ is $\tilde{w}/\hat{A}$-closed.*

(3) *Every maximal regular ideal of $\hat{A}$ is open in $\hat{A}$.*

PROOF. The proof of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is quite the same as the one of Theorem 14.

(3) $\Rightarrow$ (1). Let $\mathfrak{m}$ be a maximal regular ideal of $\hat{A}$ and $P = \mathfrak{m} \cap A$. Again, $\mathfrak{m} = \overline{\mathfrak{m} \cap A} = \overline{P}$, and it follows that $P \neq (0)$. Since $A$ is almost Dedekind, $A_P = R_w$ is a discrete rank one valuation domain and $P$ is a maximal ideal of $A$. Then by Proposition 6, $R_{\tilde{w}} = \hat{A}_w$ and by Proposition 10, $R_{\tilde{w}} = \hat{A}_{[P(\tilde{w})]}$. The rest of this proof is the same as the one of Theorem 14.

## REFERENCES


1. M. B. Boisen and M. D. Larsen, *Prüfer and valuation rings with zero divisors*, Pacific J. Math. **40** (1972), 7–12. MR **46** #9025.

2. N. Bourbaki, *General topology*. Parts 2, 3, Hermann, Paris; Addison-Wesley, Reading, Mass., 1966. MR **34** #5044.

3. M. Fukawa, *On the theory of valuations*, J. Fac. Sci. Univ. Tokyo Sect. I **12** (1965), 57–79. MR **32** #4126.

4. M. Griffin, *Prüfer rings with zero divisors*, J. Reine Angew. Math. **239/240** (1969), 55–67. MR **41** #188.

5. _____, *Rings of Krull type*, J. Reine Angew. Math. **229** (1968), 1–27. MR **36** #3778.

6. P. H. Kelly and M. D. Larsen, *Valuation rings with zero divisors*, Proc. Amer. Math. Soc. **30** (1971), 426–430. MR **44** #2745.

7. M. D. Larsen, *A generalization of almost Dedekind domains*, J. Reine Angew. Math. **245** (1970), 119–123. MR **42** #7662.

8. M. E. Manis, *Extension of valuation theory*, Bull. Amer. Math. Soc. **73** (1967), 735–736. MR **36** #1436.

9. E. Matlis, *Torsion-free modules*, Univ. of Chicago Press, Chicago, Ill., 1972.

10. J. L. Mott, *Convex directed subgroups of a group of divisibility*, Canad. J. Math. **26** (1974), 532–542. MR **51** #468.


DEPARTMENT OF MATHEMATICS, MINING UNIVERSITY, 708 33 OSTRAVE 4, CZECHOSLOVAKIA