THE COMPLETION OF PRÜFER DOMAINS

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Abstract. The completion of Prüfer and almost Dedekind domains in topologies \( \sup \{ \mathcal{T}_w : w \in \Omega \} \), where \( \mathcal{T}_w \) is a topology induced by a valuation \( w \in \Omega \), are characterized in terms of Manis valuations.

1. Let \( A \) be an integral domain with the quotient field \( K \) and let \( \Omega \) be the family of nontrivial valuations on \( K \) which are nonnegative on \( A \). It is well known that a valuation \( w \in \Omega \) with the value group \( G_w \) defines a field topology \( \mathcal{T}_w \) in \( K \) with the sets \( U_{w,\alpha} = \{ x \in K : w(x) > \alpha \} \), \( \alpha \in G_w^+ = \{ \beta \in G_w : \beta > 0 \} \), as a base of the neighbourhoods of zero in \( K \). Since \( \mathcal{T}_w \) is a minimal topology in the ordered set of all field topologies on \( K \), the completion \( \hat{K}_w \) of \( K \) is a field and the extension \( \hat{w} \) of \( w \) on \( \hat{K}_w \) is a valuation on \( \hat{K}_w \).

In this note we consider a more general situation. Let \( \mathcal{T} \) be the supremum of the family of topologies \( \{ \mathcal{T}_w : w \in \Omega \} \). In general, the family \( \Omega \) contains nonequivalent valuations, and, in this case, the completion \( \hat{K} \) of \( K \) in \( \mathcal{T} \) may contain zero divisors. Hence, it seems natural to use for the investigation of the ideal-theoretic properties of the completion \( \hat{A} \) of a domain \( A \) in a topology \( \mathcal{T} \) the valuation theory on rings with zero divisors which was introduced by M. E. Manis [8]. Especially if \( A \) is a Prüfer domain or almost Dedekind domain, we determine to what extent these properties hold for \( \hat{A} \).

In this paper all rings and groups are assumed to be commutative. At first, we recall some results of Manis [8] and M. Griffin [4]. A valuation on a ring \( R \) is a map \( w \) from \( R \) onto a totally ordered group \( G \) and a symbol \( \infty \), such that for all \( a, b \in R \),

(i) \( w(ab) = w(a) + w(b) \),
(ii) \( w(a + b) \geq \min\{w(a), w(b)\} \).

Note that in (ii) equality holds if \( w(a) \not= w(b) \). The valuation ring \( R_w \) of \( w \) is defined to be the subring of \( R \) of all elements with nonnegative value. The set \( M(w) \) of all elements of \( R \) with positive value is a prime ideal of \( R_w \) and \( (R_w, M(w)) \) is said to be a valuation pair associated with \( w \). An ideal \( Q \) of a valuation ring \( R_w \) is \( w \)-closed if \( x \in Q, y \in R_w \) and \( w(x) \leq w(y) \) imply \( y \in Q \). M. Griffin [4] defines the large quotient ring of a ring \( R \) with respect to a multiplicative system \( S \) of \( R \) as \( R_{[S]} = \{ x \in K : xs \in R \text{ for some } s \in S \} \), where \( K \) is the total quotient ring of \( R \). If \( P \) is a prime ideal of \( R \) with
$S = R - P, R_{[P]}$ will be used for $R_{[S]}$. Also $[P]R_{[P]}$ denotes the prime ideal of $R_{[P]}$ defined by $\{x \in K : xs \in P$ for some $s \in R - P\}$. Then a Prüfer ring is a ring in which every finitely generated regular ideal is invertible. By [4, Theorem 13], $R$ is a Prüfer ring if and only if for every maximal regular ideal $Q$ of $R, (R_{[Q]}, [Q]R_{[Q]})$ is a valuation pair associated with some valuation on $K$.

M. D. Larsen [7] introduced the notion of an $N$-ring which generalizes almost Dedekind domain, where an almost Dedekind domain is an integral domain $A$ such that for all prime ideals $P$ of $A, A_p$ is a discrete rank one valuation ring. He defines an $N$-ring to be a ring $R$ in which for all maximal regular prime ideals $Q$ of $R, (R_{[Q]}, [Q]R_{[Q]})$ is a valuation pair associated with a valuation $w$ such that $G_w$ is isomorphic to the group $Z$ of integers.

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2. We begin this section with some results about the topology induced by a Manis valuation. Let $(R_w, M(w))$ be a valuation pair and let $K$ be the total quotient ring of $R_w$. Then the sets

$$U_{w, \alpha} = \{x \in K : w(x) > \alpha\}, \quad \alpha \in G_w^+,$$

form a base of the neighbourhoods of zero in $K$ for some ring topology $\overline{\tau}_w$ in $K$. For, $U_{w, \alpha}$ is an additive subgroup of $K$, $U_{w, \alpha} \cdot U_{w, \beta} \subseteq U_{w, \alpha}$, and for any $x_0 \in K, \alpha \in G_w^+$, we have $x_0 U_{w, \beta} \subseteq U_{w, \alpha}$, where $\beta \geq \alpha - w(x_0)$ if $x_0 \notin w^{-1}(\infty)$ and $\beta \in G_w^+$ for $x_0 \in w^{-1}(\infty)$. Now, if we define on the set $G_w' = G_w \cup \{\infty\}$ a topology by setting $X = X \cup \{\infty\}$ for every $X \subseteq G_w', X \neq \emptyset, \emptyset$, then $G_w'$ is a complete uniform space and $w : (K, \overline{\tau}_w) \to G_w'$ is continuous.

**Lemma 1.** Let $R_w$ be a valuation ring with $J$ an ideal of $R_w$. If $J$ is $w$-closed and $J \neq w^{-1}(\infty)$, it is clopen in $R_w$. If $J$ is closed, then $w^{-1}(\infty) \subseteq J$.

**Proof.** Let $J$ be $w$-closed Since $J \neq w^{-1}(\infty)$, there exists $a \in J - w^{-1}(\infty)$. Then for every $x \in J$ and $y \in U_{w, \bar{a}}$ we have either $w(x + y) > w(x)$ or $w(x + y) > w(a)$, hence $x + U_{w, \bar{a}} \subseteq J$ and $J$ is open in $R_w$. Since $J$ is a subgroup of $R_w, J$ is clopen. Now let $J$ be closed in $R_w$ and let $x \in w^{-1}(\infty)$. Then $x \in \overline{\{0\}}$, the closure of $\{0\}$ in $R_w$, and $\{0\} \subseteq J$.

**Lemma 2.** A ring $K$ contains no zero divisors if there exists a nontrivial valuation on $K$ such that the associated topology is Hausdorff.

**Proof.** Suppose that there exists a nontrivial valuation $w$ on $K$ such that $(K, \overline{\tau}_w)$ is Hausdorff. Then $w^{-1}(\infty) \subseteq \{0\} = \{0\}$. If there is a zero divisor in $K$, then there exists $x \in w^{-1}(\infty), x \neq 0$, which cannot happen.

Further, suppose that $A$ is an integral domain with the quotient field $K$ and let $\Omega$ be the family of nontrivial valuations on $K$ which are nonnegative on $A$ and put

$$\overline{\Omega} = \sup\{\overline{\tau}_w : w \in \Omega\}.$$
If \((\hat{K}_w, \hat{\mathfrak{m}}_w)\) and \((\hat{K}, \hat{\mathfrak{m}})\) are the completions of \((K, \mathfrak{m}_w)\) and \((K, \mathfrak{m})\), respectively, we denote by \(\hat{w}, \hat{\mathfrak{m}}\) the continuous extension of \(w\) on \(\hat{K}_w\) and \(\hat{K}\), respectively. It is well known that \(\hat{w}\) is a valuation on the field \(\hat{K}_w\) and \(\hat{\mathfrak{m}}_w = \mathfrak{m}_w\).

**Lemma 3.** \(\hat{w}\) is a Manis valuation on \(\hat{K}\) for any \(w \in \Omega\).

**Proof.** We obtain the result from the fact that \(K\) is a dense subset in the Hausdorff space \(\hat{K}\) and \(\hat{w}\) is continuous.

Now, since \(\hat{w}\) is a Manis valuation on \(\hat{K}\), it defines a topology \(\mathcal{T}_w\) on \(\hat{K}\). Then we can prove the following:

**Proposition 4.** \(\widehat{\mathcal{T}} = \sup(\mathcal{T}_w : w \in \Omega)\).

**Proof.** We consider the following diagram

\[
\begin{array}{ccc}
K & \xrightarrow{f_w} & K_w = (K, \mathcal{T}_w) \\
\downarrow{i} & & \downarrow{i_w} \\
\hat{K} & \xrightarrow{\hat{f}_w} & \hat{K}_w \\
\sim{w} & & \hat{w} \\
\end{array}
\]

where \(w \in \Omega\), \(\hat{f}_w\) is the continuous extension of \(f_w = \text{id}_K\) and \(i\), \(i_w\) are the canonical injections. We have \(\hat{w} \cdot i = w, \hat{f}_w \cdot i = i_w \cdot f_w, w = \hat{w} \cdot i_w \cdot f_w\); thus \(\hat{w} \cdot i = \hat{w} \cdot \hat{f}_w \cdot i\). Since \(i(K)\) is a dense subset in the Hausdorff space \(\hat{K}\), we obtain \(\hat{w} = \hat{w} \cdot \hat{f}_w\). Now, since \(U_{w,\alpha} = \hat{f}_w^{-1}(U_{w,\alpha})\) for every \(\alpha \in G_w^+, w \in \Omega\), \(U_{w,\alpha}\) is open in \(K_w\); it follows that \(U_{w,\alpha}\) is open in \(\hat{K}_w\); thus \(\widehat{\mathcal{T}} \supset \mathcal{T}_w\) for every \(w \in \Omega\). Suppose that \(\mathcal{T}\) is a topology on \(\hat{K}\) such that \(\mathcal{T} \supset \mathcal{T}_w\) for every \(w \in \Omega\). Then from the fact that \(\hat{w} = \hat{w} \cdot \hat{f}_w\), we obtain that \(\hat{f}_w\) is continuous in \((\hat{K}, \mathcal{T})\) for every \(w \in \Omega\). Thus, using the fact \(\widehat{\mathcal{T}} = \sup(\mathcal{T}_w : w \in \Omega)\), we obtain by [2, §3, Proposition 18], that the topology \(\widehat{\mathcal{T}}\) is the supremum of topologies on \(\hat{K}\) for which \(\hat{f}_w\), \(w \in \Omega\), are continuous. Hence, \(\widehat{\mathcal{T}} \leq \mathcal{T}\) and \(\widehat{\mathcal{T}} = \sup(\mathcal{T}_w : w \in \Omega)\).

Henceforth, we assume that \(K\) is a subring of \(\hat{K}\).

**Proposition 5.** Let \(w \in \Omega\) and let \((R_w, M(w))\) be the valuation pair associated with \(\hat{w}\). Let \(R_w\) be a valuation ring with the maximal ideal \(M(w)\) of a valuation \(w\) in a field \(K\). Then

\[
R_w = \overline{R_w}, \quad M(\hat{w}) = \overline{M(w)},
\]

where \(X\) is the closure of \(X\) in \((\hat{K}, \mathcal{T})\).

**Proof.** Let \(x \in \hat{K} - \overline{R_w}\). Then by Proposition 4, there exist \(w_1, \ldots, w_n \in \Omega, \alpha_i \in G_w^+\), \(i = 1, \ldots, n\), such that
There is no loss of generality in assuming that \( w \in \{ w_1, \ldots, w_n \} \), say \( w = w_1 \). Moreover, since \( K \) is a dense subset in \( \hat{K} \), there exists \( z \in K \) such that

\[
\hat{w}_1(z - x) > \alpha_i, \quad i = 1, \ldots, n.
\]

Hence, \( z \in K - R_w \) and \( w(z) = w_1(z) < 0 \). Suppose that \( w_1(z) \neq \hat{w}_1(x) \). Then

\[
\alpha_1 < \hat{w}_1(z - x) = \min \{ \hat{w}_1(x), w_1(z) \} < 0,
\]
a contradiction with \( \alpha_1 > 0 \). Thus, \( \hat{w}(x) = w(z) < 0 \) and \( x \notin R_w \). Therefore, \( R_w \subseteq \hat{R}_w \). On the other hand, \( R_w \) is closed in \( \hat{K} \) and we have \( \hat{R}_w \subseteq R_w \). Thus, \( R_w = \hat{R}_w \). The rest can be proved analogously.

For future investigation we need to solve the following problem. If \( w \) is a valuation on the quotient field \( K \) of \( A \) such that \( R_w = A_{P(w)} \), where \( P(w) = M(w) \cap A \), when \( R_w = (\hat{A}_w)_{P(\hat{w})} \) holds, where \( \hat{A}_w \) is the closure of \( A \) in \( \hat{K} \) and \( P(\hat{w}) = M(\hat{w}) \cap \hat{A}_w \). The following example shows that it does not hold in general.

**Example.** Let \( Q \) be the field of rational numbers and let \( X \) be an indeterminate over \( Q \). In the field \( Q(X) \) we define a valuation \( w \) in the following way.

\[
w \left( \sum_{i=0}^{n} a_i X^i \right) = k \Leftrightarrow a_0 = \cdots = a_{k-1} = 0, \quad a_k \neq 0;
\]

\[
w(f/g) = w(f) - w(g), \quad f, g \in Q[X], \quad g \neq 0, \quad w(0) = \infty.
\]

Then \( R_w \) is a discrete rank one valuation ring and \( R_w = Z[X]_{M(w) \cap Z[X]} \). Now in the quotient field \( Q((X)) \) of the ring \( Q[[X]] \) of formal power series over \( Q \), we define a valuation \( \hat{w} \) in the following way.

\[
\hat{w} \left( \sum_{n=0}^{\infty} a_n X^n \right) = k \Leftrightarrow a_0 = \cdots = a_{k-1} = 0, \quad a_k \neq 0, \quad a_n \in Q;
\]

\[
\hat{w}(f/g) = \hat{w}(f) - \hat{w}(g), \quad f, g \in Q[[X]], \quad g \neq 0, \quad \hat{w}(0) = \infty.
\]

Then \( (Q((X)), \hat{R}_w) \) is the completion of \( (Q(X), R_w) \), \( \hat{w} \) is the unique continuous extension of \( w \) on \( Q((X)) \) and \( Z[[X]] \) is the closure of \( Z[X] \) in \( Q((X)) \). To show that \( R_w \neq Z[[X]]_{M(\hat{w}) \cap Z[[X]]} \), we need a simple lemma.

**Lemma.** Let \( n \) be a natural number, \( z_i \in Z^+, \ i = 0, \ldots, n-1 \), be such that \( 0 \leq z_i < p_{n-i} \), where \( p_i \) is the \( i \)-th prime number and let \( z_0(1/p_n) + z_1(1/p_{n-1}) + \cdots + z_{n-1}(1/2) \in \mathbb{Z} \). Then \( z_0 = z_1 = \cdots = z_{n-1} = 0 \).

**Proof.** The proof is by induction on \( n \). The lemma holds for \( n = 1 \). We assume that the lemma holds for \( n \). Let \( z_i \in \mathbb{Z}, \ i = 0, \ldots, n \), be such that \( 0 \leq z_i < p_{n+1-i} \), and let \( z_0(1/p_n) + \cdots + z_n(1/2) \in \mathbb{Z} \). Let \( z'_i, \ i = \)
0, \ldots, n - 1, be such that
\[ z_{i+1} \equiv z'_i \pmod{p_{n-i}}, \quad 0 \leq z'_i < p_{n-i}. \]
Then \( z'_0(1/p_n) + \cdots + z'_{n-1}(1/2) \in Z, \) and by induction we have \( z'_0 = \cdots = z'_{n-1} = 0. \) Hence, \( z_1 p_{n+1}/p_n \in Z, \ldots, z_n p_{n+1}/2 \in Z, \) and we obtain \( z_0 = z_1 = \cdots = z_n = 0. \)

Now, \( f = 1 + (1/2)X + \cdots + (1/p_n)X^n + \cdots \in R_\infty. \) We suppose that there exist \( u = \sum a_n X^n, \ v = \sum b_n X^n \in Z[[X]] \) such that \( a_0 \neq 0 \) and \( v = f \cdot u. \) Then for any natural \( n \) we have
\[ b_n = a_n + (1/2)a_{n-1} + \cdots + (1/p_n)a_0. \]
Let \( a_{n-i} = z_{n-i} \pmod{p_i}, \ i = 1, \ldots, n, \ 0 \leq z_{n-i} < p_i. \) Then \( z_0(1/p_n) + \cdots + z_{n-1}(1/2) \in Z, \) and by the lemma we have \( z_0 = \cdots = z_{n-1} = 0. \) Thus, \( p_n \) divides \( a_0 \) for every \( n, \) a contradiction.

The partial solution of the problem gives the following two propositions.

**Proposition 6.** Let \( w \) be a discrete rank one valuation on the quotient field \( K \) of \( A \) such that \( R_w = A_{P(w)} \) and let \( P(w) \) be a maximal ideal of \( A. \) Then \( R_w = \hat{A}_w. \)

**Proof.** It is well known that in this case \( \mathfrak{g}_w \) is the \( M(w) \)-adic topology. Then \( \hat{A}_w \) is the completion of \( A \) in \( P(w) = M(w) \cap A \)-adic topology and, since \( P(w) \) is maximal in \( A, \) \( \hat{A}_w \) is local with the unique maximal ideal \( P(\hat{w}) = M(\hat{w}) \cap \hat{A}_w. \) Hence, \( R_w = A_{P(w)} \subseteq (\hat{A}_w)_{P(\hat{w})} = \hat{A}_w \) and \( R_w = \hat{R}_w \subseteq \hat{A}_w, \) where the vinculum denotes the closure in \( \hat{K}_w. \) Therefore, \( R_w = \hat{A}_w. \)

Using the example it should be observed that “\( P(w) \) is maximal” cannot be removed from Proposition 6.

The following proposition follows immediately from [10, Theorem 2.1].

**Proposition 7.** \( R_w = (\hat{A}_w)_{P(\hat{w})} \) if and only if there exists an order epimorphism \( f \) from a group of divisibility \( G(\hat{A}_w) \) of \( \hat{A}_w \) onto \( G_w \) such that \( f \cdot v = \hat{w}, \) where \( v \) is the canonical map from \( K^* \) onto \( G(\hat{A}_w). \)

Further, if \( R_w = (\hat{A}_w)_{P(\hat{w})} \) for a valuation \( w, \) it is natural to ask when \( R_w = \hat{A}_w \) for \( P(w) = M(\hat{w}) \cap \hat{A}_w. \) To solve this problem, we prove two lemmas.

At first, on the family \( \Omega \) we may define an equivalence relation
\[ w \equiv w' \quad \text{if and only if} \quad w, w' \text{ are dependent.} \]
Let \( \Omega_0 \) be a family of representatives of the equivalence classes.

**Lemma 8.** Let for every \( w \in \Omega_0, x_w \) be an element from \( \hat{K}_w. \) Then there exists \( x \in \hat{K} \) such that \( \hat{f}_w(x) = x_w \) for every \( w \in \Omega_0. \) Furthermore, if \( x_w \neq 0 \) for every \( w \in \Omega_0, \) \( x \) is regular.

**Proof.** We set \( \Delta = \prod(i_w: w \in \Omega_0), \) where \( i_w: K \to \hat{K}_w \) is the canonical map. Then using the approximation theorem for independent valuations, we obtain that
$\Delta: K \to \prod \{ \hat{K}_w: w \in \Omega_0 \}$

is a completion of the topological field $(K, \mathbb{K})$, and the canonical projection map $\text{pr}_w$ of $\Pi(\hat{K}_w: w \in \Omega_0)$ onto $\hat{K}_w$ is the unique continuous homomorphism such that

$$i_w \cdot f_w = \text{pr}_w \cdot \Delta, \quad w \in \Omega_0.$$ 

Now, since $i: K \to \hat{K}$ is the completion of $K$, there exists an isomorphism $\varphi$ such that $\varphi \cdot \Delta = i$. Then $(\hat{f}_w \cdot \varphi) \cdot \Delta = \hat{f}_w \cdot i = i_w \cdot f_w$, hence $\text{pr}_w = \hat{f}_w \cdot \varphi$ for every $w \in \Omega_0$. Let $(x_w) \in \Pi \{ \hat{K}_w: w \in \Omega_0 \}$. We set $x = \varphi((x_w)) \in \hat{K}$. Then $\hat{f}_w(x) = \text{pr}_w((x_w)) = x_w$ and if $x_w \neq 0$ for every $w \in \Omega_0$, $(x_w)$ is regular in $\Pi(\hat{K}_w: w \in \Omega_0)$. Thus, $x$ is regular in $\hat{K}$.

For the second lemma we need some notation from \cite{5}. Let $w, w' \in \Omega$. If $R_w \subseteq R_w'$ we say that $w'$ is coarser than $w$ and write $w' < w$. If $w' < w$ then $G_w = G_w/H$, where $H$ is an isolated subgroup of $G_w$. Since the valuations coarser than $w$ are totally ordered there will be a finest valuation $w'' = w' \wedge w$ coarser than both $w, w'$. Let $H$ be a corresponding subgroup of $G_w$. Then we denote by $(w,w')\overline{\alpha}$ the element $\psi(\alpha)$, where $\psi: G_w \to G_w$ is the canonical map and $\alpha \in G_w$. Now, let $w_1, \ldots, w_n \in \Omega$. Let $(a_1, \ldots, a_n) \in G_{w_1} \times \cdots \times G_{w_n}$, $b_i \in K, 1 \leq i \leq n$. We define $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ to be concordant when the following conditions hold:

1. If $(a_i)\overline{\alpha} = (a_j)\overline{\alpha}$ then $(a_i)\overline{\alpha} \leq (a_j)\overline{\alpha}$.
2. If $(a_i)\overline{\alpha} > (a_j)\overline{\alpha}$ then $(a_i)\overline{\alpha} = (a_j)\overline{\alpha} (b_i - b_j)$.

where $(w,w')\overline{\alpha} = (w,w')\overline{\alpha}$. Then we say that $\Omega$ satisfies the weak reinforced approximation theorem (W.R.A.T.) for $A$ if, for any finite number $w_i \in \Omega$, $i = 1, \ldots, n$, of valuations with $(\alpha_1, \ldots, \alpha_n) \in G_{w_1}^+ \times \cdots \times G_{w_n}^+$ and $(b_1, \ldots, b_n) \in \mathbb{A}$ such that $((\alpha_1, \ldots, \alpha_n), (b_1, \ldots, b_n))$ is concordant, there exists $\alpha \in A$ such that

$$w_i(\alpha - b_i) = \alpha_i \quad i = 1, \ldots, n.$$ 

By \cite[Proposition 24]{5}, for every Prüfer domain $A$, $\Omega$ satisfies the W.R.A.T.

**LEMMA 9.** Let $\Omega$ satisfy the W.R.A.T. for $A$. Then

$$\hat{A} = \cap \{ \hat{f}_w^{-1}(\hat{A}_w): w \in \Omega_0 \}.$$ 

**PROOF.** At first, since $\hat{f}_w$ is the continuous extension of $f_w = \text{id}_K$ for every $w$, we have $\hat{f}_w(\hat{A}) \subseteq \hat{f}_w(A) = \hat{A}_w$. Conversely, let $x \in \hat{f}_w^{-1}(\hat{A}_w)$ for every $w \in \Omega_0$. Let $w_i \in \Omega_0, i = 1, \ldots, n$, $\alpha_i \in G_{w_i}^+$. Then since $A$ is a dense subset in $\hat{A}_w$, for every $i, 1 \leq i \leq n$, we may find an element $y_i$ such that

$$w_i(y_i) \geq \alpha_i, \quad i = 1, \ldots, n.$$ 

Now, since $w_i$ are pairwise independent, $((\alpha_1, \ldots, \alpha_n), (y_1, \ldots, y_n))$ is concordant. Thus, there exists $\alpha \in A$ such that $w_i(\alpha - y_i) \geq \alpha_i, \quad i = 1, \ldots, n$.

Then using the identity $\hat{w}_i = w_i \cdot \hat{f}_w$ from the proof of Proposition 4, we have
\[ \hat{w}_i(a - x) = \hat{w}_i \cdot \hat{f}_{w_i}(a - x) = \hat{w}_i \left( a - \hat{f}_{w_i}(x) \right) \]
\[ = \hat{w}_i \left( a - y_i + y_i - \hat{f}_{w_i}(x) \right) > \alpha_i, \quad 1 \leq i \leq n. \]

Therefore, \( A \) is a dense subset in \( \bigcap \{ \hat{f}_{w_i}^{-1}(\hat{A}_w) : w \in \Omega_0 \} \).

**Proposition 10.** Let \( w \in \Omega \) be such that \( R_w = \hat{A}_{\{P(\hat{w})\}} \). Then \( R_w = (\hat{A}_w)_{P(\hat{w})} \).

Conversely, if \( \Omega \) satisfies the W.R.A.T. for \( A \) and if \( R_w = (\hat{A}_w)_{P(\hat{w})} \), then
\[ R_w = \hat{A}_{\{P(\hat{w})\}}, \quad M(\hat{w}) = [P(\hat{w})] \hat{A}_{\{P(\hat{w})\}}. \]

**Proof.** Let \( R_w = \hat{A}_{\{P(\hat{w})\}} \) and let \( x \in R_w \). Then for \( y \in \hat{f}_{w_s}^{-1}(x) \) there exists \( z \in \hat{A} - P(\hat{w}) \) such that \( y \cdot z \in \hat{A} \). Hence, \( x \cdot \hat{f}_{w_s}(z) \in \hat{A}_w \) and \( \hat{f}_{w_s}(z) \in \hat{A}_w - P(\hat{w}) \). Thus, \( x \in (\hat{A}_w)_{P(\hat{w})} \). The converse inclusion is trivial. Conversely, let \( R_w = (\hat{A}_w)_{P(\hat{w})} \). Let \( \Omega_0 \) be the family of representatives of the equivalence classes of \( \equiv \) such that \( w \in \Omega_0 \). Let \( x \in R_w \). Then there exists \( y_w \in \hat{A}_w - P(\hat{w}) \) such that \( y_w \cdot \hat{f}_{w_s}(x) \in \hat{A}_w \). Further, since \( \hat{K}_w \) is the quotient field of a domain \( \hat{A}_w \) for every \( w' \in \Omega_0 \), there exist \( y_{w'} \in \hat{A}_{w'}, w' \in \Omega_0, w' \neq w \), such that \( y_w \cdot \hat{f}_{w_s}(x) \in \hat{A}_{w'} \). Now, by Lemma 8, there exists \( y \in \hat{K} \) such that \( \hat{f}_{w_s}(y) = y_{w'}, w' \in \Omega_0 \). Then by Lemma 9, \( y \in \hat{A} \), and using the identity \( \hat{w} = \hat{w} \cdot \hat{f}_{w_s}, y \in \hat{A} - P(\hat{w}) \). Further,
\[ \hat{f}_{w_s}(y \cdot x) = y_w \cdot \hat{f}_{w_s}(x) \in \hat{A}_{w'}, \quad w' \in \Omega_0, \]
hence \( y \cdot x \in \hat{A} \). Therefore, \( x \in \hat{A}_{\{P(\hat{w})\}} \). The converse inclusion is trivial.

Further, it is clear that \( [P(\hat{w})] \hat{A}_{\{P(\hat{w})\}} \subseteq M(\hat{w}) \). Let \( y \in M(\hat{w}) \subseteq \hat{A}_{\{P(\hat{w})\}} \). Then there exists \( z \in \hat{A} - P(\hat{w}) \) such that \( y \cdot z \in \hat{A} \) and it is easy to see that \( y \cdot z \in P(\hat{w}) \). Thus, \( [P(\hat{w})] \hat{A}_{\{P(\hat{w})\}} = M(\hat{w}) \).

**Lemma 11.** Let \( \Omega \) satisfy the W.R.A.T. for \( A \). Then \( \hat{A} = \bigcap \{ R_w : w \in \Omega_0 \} \) if and only if \( A \) is a dense subset in \( (R_w, \bigcap_w R_w) \) for every \( w \in \Omega_0 \).

**Proof.** If \( A \) is a dense subset in \( R_w \), then \( \hat{A}_w = R_w \) and the rest follows by Lemma 9. Conversely, let \( \hat{A} = \bigcap \{ R_w : w \in \Omega_0 \} \). If \( A \) is not a dense subset in \( R_w \) for some \( w \in \Omega_0 \), then \( \hat{A}_w \neq R_w \), and there exists \( x_w \in \hat{A}_w \) such that \( \hat{f}_{w_s}(x_w) \in \hat{A}_w \). Further, let for every \( w' \in \Omega_0, w' \neq w, x_w \in R_{w'} \). By Lemma 8, there exists \( x \in \hat{K} \) such that \( \hat{f}_{w_s}(x) = x_w \) for every \( w' \in \Omega_0 \). Thus, \( x \in \hat{f}_{w_s}^{-1}(R_{w'}) = R_w, w' \in \Omega_0 \). Hence, \( x \in \hat{A} \) and \( x_w \in \hat{A}_w \), a contradiction.

**Theorem 12.** Let \( A \) be a Prüfer domain such that there exists a family \( \Omega_0 \) of representatives of the equivalence classes of \( \equiv \) such that \( A \) is a dense subset in \( (R_w, \bigcap_w R_w) \) for every \( w \in \Omega_0 \). Then \( \hat{A} \) is a Prüfer ring.

**Proof.** By Lemma 11, \( \hat{A} = \bigcap \{ R_w : w \in \Omega_0 \} \). Let \( a, b \in \hat{A} \) be such that the ideal \( (a, b) \) in \( \hat{A} \) is regular. Since \( R_w \) is a valuation domain, for every \( w \in \Omega_0 \) there exists \( x_w \in R_w \) such that \( (\hat{f}_{w_s}(a), \hat{f}_{w_s}(b)) R_w = x_w R_w \). By Lemma 8, there exists \( x \in \hat{K} \) such that \( \hat{f}_{w_s}(x) = x_w \) for every \( w \in \Omega_0 \). Hence \( x \in R_w \) for every \( w \in \Omega_0 \) and \( x \in \hat{A} \). Then using the identity \( \text{pr}_w = \hat{f}_{w_s} \cdot \varphi \) from the proof of Lemma 8, we obtain \( (a, b) = x \hat{A} \), where \( x \) is regular. Thus \( (a, b) \) is invertible and, by the definition, \( \hat{A} \) is a Prüfer ring.
It should be observed that there is a reasonably large class of Prüfer domains which satisfy the conditions of Theorem 12. In fact, every \( h \)-local Prüfer domain is of such type. Recall that an integral domain \( A \) is said to be \( h \)-local if every nonzero ideal of \( A \) is contained in only a finite number of maximal ideals, and if every nonzero prime ideal of \( A \) is contained in only one maximal ideal. Then by [9, Theorem 22], \( A \) is \( h \)-local if and only if \( A = \prod \{ A_M : M \in \text{mspec } A \} \), where \( \text{mspec } A \) is the set of maximal ideals of \( A \), \( \tilde{A} \) is the completion of \( A \) in the \( A \)-topology, and \( \tilde{A}_M \) is the completion of \( A_M \) in the \( A_M \)-topology, where for any ring \( R \) the \( R \)-topology is the topology on \( R \) with the ideals \( rR \), \( r \in R^* \), being a subbase for the open neighbourhoods of 0 in \( R \).

The following lemma holds.

**Lemma 13.** Let \( A \) be a \( h \)-local Prüfer domain. Then the topology \( \sup \{ \mathfrak{F}_w : w \in \Omega \} \) on \( A \) is the same as the \( A \)-topology.

**Proof.** Let \( w_i \in \Omega \), \( \alpha_i \in G^+_w \). Then there exist \( a_i \in A \), \( i = 1, \ldots, n \), such that \( w_i(a_i) > \alpha_i \). Then

\[
a_1A \cap \cdots \cap a_nA \subseteq \bigcap_{i=1}^n U_{w_i, \alpha_i}.
\]

Conversely, let \( a \in A^* \). Then since \( A \) is \( h \)-local, there exist only a finite number of maximal ideals \( M_1, \ldots, M_n \in \text{mspec } A \) such that \( a \in M_i \). Let \( w_i \in \Omega \), \( i = 1, \ldots, n \), be such that \( M_i = P(w_i) \), and we set \( \alpha_i = w_i(a) \). Let \( x \in A \cap \bigcap_{i=1}^n U_{w_i, \alpha_i} \). If \( x \notin aA \), then there exists \( w \in \Omega \) such that \( w(\alpha^{-1}) < 0 \). Since \( w \neq w_i \), we have \( a \notin P(w) \) and \( w(\alpha^{-1}) = w(x) > 0 \), a contradiction. Thus, \( \bigcap_{i=1}^n U_{w_i, \alpha_i} \cap A \subseteq aA \) and \( \sup \{ \mathfrak{F}_w \} \) on \( A \) is the \( A \)-topology.

Since for every \( h \)-local Prüfer domain \( A \), the family \( \Omega_0 = \{ \langle \omega \in \Omega : P(\omega) \in \text{mspec } A \} \) is a set of representatives of the equivalence classes of \( \equiv \), \( A \) satisfies the conditions of Theorem 12 and \( \tilde{A} \) is a Prüfer ring.

Further, we say that a ring \( A \) with the total quotient ring \( K \) is a \( \tilde{A} \)-Prüfer ring, where \( \tilde{A} \) is a ring topology on \( K \) if, for every maximal regular ideal \( m \) of \( A \), \( (A_m|P|A|_m) \) is a valuation pair associated with a valuation \( w \) on \( K \) such that \( w \) is continuous in \( \tilde{A} \).

To generalize the definition of \( w \)-closed ideals, we say that an ideal \( Q \) of \( A \) is \( \tilde{w} / A \)-closed for some \( w \in \Omega \) if \( x \in Q \), \( y \in A \), and \( \tilde{w}(x) < \tilde{w}(y) \) imply \( y \in Q \).

We are now ready to prove the main theorem.

**Theorem 14.** Let \( A \) be a Prüfer domain. Then the following conditions are equivalent.

1. \( \tilde{A} \) is a \( \tilde{\tilde{A}} \)-Prüfer ring.
2. For every maximal regular ideal \( m \) of \( \tilde{A} \) there exists \( w \in \Omega \) such that \( m \) is \( \tilde{w} / A \)-closed and \( R_w = (\tilde{A}_w)_{P(\tilde{w})} \) for every \( w \in \Omega \).
3. Every maximal regular ideal of \( \tilde{A} \) is open in \( \tilde{A} \) and \( R_w = (\tilde{A}_w)_{P(\tilde{w})} \) for every \( w \in \Omega \).
Proof. (1) \(\Rightarrow\) (2). Let \(m\) be a maximal regular ideal of \(\hat{A}\). Then \((\hat{A}_{[m]}, [m]/\hat{A}_{[m]})\) is a valuation pair associated with a valuation \(v\) continuous in \(\mathcal{F}\). Let \(w = v|K\). It is easy to see that \(w\) is a valuation on \(K\). If \(w\) is the trivial one, it follows since \(w\) is continuous in \(\mathcal{F}\), that \(\mathcal{F}\) is discrete. Thus, \(K = K, \hat{A} = A, m = M(w) \cap A = (0)\), a contradiction with \(m\) regular. Hence \(w\) is nontrivial, \(w \in \Omega, \hat{w} \text{ and } v\) are the continuous extensions of \(w\) on \(K\). Since \(K\) is Hausdorff, we obtain \(\hat{w} = v\). Hence \(R_{\hat{w}} = \hat{A}_{[m]}, M(\hat{w}) = [m]\hat{A}_{[m]}\) and it is easy to see that \(\hat{A}_{[m]} = \hat{A}_{[P(\hat{w})]}, P(\hat{w}) = m\). Thus, \(m\) is \(\hat{w}/\hat{A}\)-closed and the rest follows by Proposition 10.

(2) \(\Rightarrow\) (3). Let \(m\) be a maximal regular ideal of \(\hat{A}\) and let \(x \in m\) be regular. Then for \(z \in U_{\hat{w}, \hat{w}(x)} \cap \hat{A}\) we have \(\hat{w}(y + z) \geq \hat{w}(z) \geq \hat{w}(x)\) if \(\hat{w}(y) \geq \hat{w}(z)\) and \(\hat{w}(y + z) = \hat{w}(y)\) if \(\hat{w}(y) < \hat{w}(z)\). Hence \(y + z \in m\) and \(m\) is open in \(\hat{A}\).

(3) \(\Rightarrow\) (1). Let \(m\) be a maximal regular ideal of \(\hat{A}\) and \(P = A \cap m\). Since \(A\) is a dense subset in \(\hat{A}\), and \(m\) is open in \(\hat{A}\), it follows that \(m = m \cap A = P\), where the vinculum denotes the closure in \(\hat{A}\). Since \(m\) is regular, we have \(P \neq (0)\). Let \(w \in \Omega\) be such that \(R_w = A_P\). By Proposition 10, \(R_{\hat{w}} = \hat{A}_{[P(\hat{w})]}\). It is clear that \(P \subseteq P(\hat{w})\). Suppose that there exists \(x \in P(\hat{w}) - P\); then for some \(w_1, \ldots, w_n \in \Omega, \alpha_i \in G_{w_i}^+, i = 1, \ldots, n\), we have

\[
\left( x + \bigcap_{i=1}^n U_{w_i, \alpha_i} \right) \cap P = \emptyset.
\]

On the other hand, there exists \(a \in A\) such that

\[
\hat{w}_i(a - x) > \alpha_i, \quad i = 1, \ldots, n.
\]

Again, there is no loss of generality in assuming that \(w \in \{w_1, \ldots, w_n\}\), say \(w = w_1\). Then since \(P = P(\hat{w})\), it follows that

\[
\alpha_1 < \hat{w}_1(a - x) = \min\{w_1(a), \hat{w}_1(x)\} = 0,
\]

a contradiction. Thus, \(P(\hat{w}) = P = m\), and \((\hat{A}_{[m]}, [m]\hat{A}_{[m]})\) is a valuation pair associated with the valuation \(w\) continuous in \(\mathcal{F}\). Therefore, \(\hat{A}\) is a \(\mathcal{F}\)-Prüfer ring.

We have not been able to show that there exists a Prüfer domain \(A\) such that \(\hat{A}\) is not a \(\mathcal{F}\)-Prüfer ring, and we do not know if there is a Prüfer domain \(A\) such that \(\hat{A}\) is a Prüfer ring but \(\hat{A}\) is not a \(\mathcal{F}\)-Prüfer ring.

Finally, we say that a ring \(A\) with the total quotient ring \(K\) is a \(\mathcal{F}\)-ring for some ring topology \(\mathcal{F}\) on \(K\), if for every maximal regular ideal \(m\) of \(A\), \((A_{[m]}, [m]A_{[m]})\) is a valuation pair associated with a valuation \(w\) on \(K\) continuous in \(\mathcal{F}\) and such that \(G_w = Z\).

Theorem 15. Let \(A\) be an almost Dedekind domain. Then the following conditions are equivalent.

(1) \(\hat{A}\) is a \(\mathcal{F}\)-ring.

(2) For every maximal regular ideal \(m\) of \(\hat{A}\) there exists \(w \in \Omega\) such that \(m\) is \(\hat{w}/\hat{A}\)-closed.

(3) Every maximal regular ideal of \(\hat{A}\) is open in \(\hat{A}\).
**Proof.** The proof of implications (1) ⇒ (2) ⇒ (3) is quite the same as the one of Theorem 14.

(3) ⇒ (1). Let \( m \) be a maximal regular ideal of \( \hat{A} \) and \( P = m \cap A \). Again, \( m = m \cap A = \hat{P} \), and it follows that \( P \neq (0) \). Since \( A \) is almost Dedekind, \( A_P = R_\omega \) is a discrete rank one valuation domain and \( P \) is a maximal ideal of \( A \). Then by Proposition 6, \( R_\omega = \hat{A}_\omega \) and by Proposition 10, \( R_\omega = \hat{A}_{[P(\omega)]} \). The rest of this proof is the same as the one of Theorem 14.

**References**


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