REPRODUCING KERNELS FOR q-JACOBI POLYNOMIALS\textsuperscript{1}

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ABSTRACT. We derive a family of reproducing kernels for the q-Jacobi polynomials $\Phi_n^{(a,\beta)}(x) = \frac{\Gamma(q^{-n}, q^{-n+1}; q; q^x)}{\Gamma(q^{-n}; q, q)}$. This is achieved by proving that the polynomials $\Phi_n^{(a,\beta)}(x)$ satisfy a discrete Fredholm integral equation of the second kind with a positive symmetric kernel, then applying Mercer's theorem.

1. Introduction. The purpose of the present note is to construct a family of reproducing kernels or bilinear formulas

$$\sum_{n=0}^{\infty} \theta_n^{(\beta)}(a,\beta)(x) \Phi_n^{(a,\beta)}(y) = K(x, y)$$

for the q-Jacobi polynomials $\{\Phi_n^{(a,\beta)}(x)\}_{n=0}^{\infty}$. For definitions and notations, see §2. These reproducing kernels are obtained by finding a linear integral operator that maps a q-Jacobi polynomial to another q-Jacobi polynomial of the same degree but with different parameters. This integral operator is the q-fractional integral $L^{(a,n)}$ of (2.9). The results obtained below are q-analogues of Ismail's results in [5]. Actually Al-Salam and Ismail [1] used certain discrete transforms, that map the Charlier and Meixner polynomials to Laguerre polynomials, to derive several bilinear formulas for the Charlier and Meixner polynomials. Later Ismail [5] modified these transforms and obtained similar formulas for the Hahn polynomials. Related results were also obtained by Rahman [6], [7] by using a completely different approach.

In the next section we define the q-Jacobi polynomials and a q-fractional integral operator. §3 contains our main results, the family of bilinear formulas (3.13). The last section, §4, is devoted to proving the square integrability of the kernel $K(a', q'; \alpha, \beta, \eta; q)$ of (3.6) with respect to the measure $\mu(x, y)$ of (3.9) and (3.10).

For an excellent survey of the theory of reproducing kernels we refer the interested reader to Hille [4].

2. Preliminaries. Throughout this work we shall always assume that $0 < q < 1$. The symbol $(a; q)_\infty$ shall stand for the convergent infinite product

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$[1 - aq^n]$. By $(a; q)_n$, or equivalently $[1 - a]_n$, we mean
\[
(a; q)_n = [1 - a]_n = (a; q)_\infty / (aq^n; q)_\infty,
\]
so that, in particular, we have $(a; q)_0 = 1$, $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ for $n = 1, 2, 3, \ldots$.

The generalized basic or $q$-hypergeometric function, $\Phi$, is defined by
\[
\Phi(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)
= \sum_{n=0}^{\infty} \frac{(a_1; q)_n(a_2; q)_n \cdots (a_r; q)_n}{(b_1; q)_n(b_2; q)_n \cdots (b_s; q)_n} \frac{z^n}{(q; q)_n}.
\]

In particular, we have the $q$-analogue of the binomial theorem [8, p. 92]
\[
\Phi_0(a; - q, z) = (az; q)_\infty / (z; q)_\infty
\]
valid for $|z| < 1$.

Let us recall the $q$-integral
\[
\int_0^x f(t) dt(q) = x(x-q) \sum_{n=0}^{\infty} q^n f(x^n).
\]

This note is concerned with what we call $q$-Jacobi polynomials as defined by
\[
\Phi_n^{(a, b)}(x) = \Phi_1(q^{-n}, q^{n+b-1}; q^a, q, qx).
\]

However we should mention here that Andrews and Askey [2] studied a more general set of polynomials, namely, $\Phi_2(q^{-n}, q^{n+a+b+1}, x; q^{a+1}, -q^{d+1}; q, q)$, which they also refer to as the $q$-Jacobi polynomial. They refer to (2.4) as the $\Phi_1$-Jacobi polynomials.

The $q$-Jacobi polynomials satisfy the orthogonality relation
\[
\int_{-\infty}^{\infty} \Phi_n^{(a, b)}(x) \Phi_m^{(a, b)}(x) d\Psi(x; a, b) = F_n(a, b) \delta_{nm}
\]
where
\[
F_n(a, b) = q^{an} \frac{(q; q)_n(q^{\beta-a}; q)_n(q^{n+b-1}; q)_n}{(q^a; q)_n(q^\beta; q)_{2n}}
\]
and $\Psi(x; a, b)$ is the step function with the jumps
\[
d\Psi(x; a, b) = \frac{(q^a; q)_\infty(q^{\beta-a}; q)_k q^{ak}}{(q^\beta; q)_\infty(q; q)_k} q^{ak}
\]
at $x = q^k$ ($k = 0, 1, 2, 3, \ldots$), $\beta > a > 0$.

(2.5) may also be written as a $q$-integral.
We shall also make use of the fractional operator $\mathcal{L}^{(\alpha, \eta)}$ for $\eta > 0$,

$$\mathcal{L}^{(\alpha, \eta)} f(x) = \int_0^1 f(x t) \, d\Psi(t; \alpha, \eta + \alpha)$$

(2.9)

$$= \frac{(q^\alpha; q)_\infty (q^{\eta}; q)_\infty}{(q^{\alpha+\eta}; q)_\infty (q; q)_\infty (1 - q)} \cdot \int_0^1 [1 - qt]_{\eta - 1} t^{\alpha - 1} f(x t) \, d(t; q).$$

In particular, using (2.2), we can show that

$$\mathcal{L}^{(\alpha, \eta)} \left\{ x^n \right\} = \frac{(q^\alpha; q)_n}{(q^{\alpha+\eta}; q)_n} x^n \quad (n = 0, 1, 2, 3, \cdots)$$

(2.10)

so that

$$\mathcal{L}^{(\alpha, \eta)} \Phi_n^{(\alpha, \beta)}(x) = \Phi_n^{(\alpha+\eta, \beta)}(x).$$

(2.11)

More explicitly, (2.11) can also be written as

$$\sum_{k=0}^\infty q^{nk} \frac{(q^\alpha; q)_\infty (q^{\eta}; q)_k}{(q^{\alpha+\eta}; q)_\infty (q; q)_k} \Phi_n^{(\alpha, \beta)}(x q^k) = \Phi_n^{(\alpha+\eta, \beta)}(x).$$

(2.12)

3. Reproducing kernels. (2.11) shows that the $q$-fractional integral operator $\mathcal{L}^{(\alpha, \beta)}$ maps a $q$-Jacobi polynomial to a $q$-Jacobi polynomial. Hence we have

$$F_n(\alpha + \eta, \beta) \delta_{nn}$$

(3.1)

$$= \int_{-\infty}^\infty \left\{ \mathcal{L}^{(\alpha, \eta)} \Phi_n^{(\alpha, \beta)}(x) \right\} \left\{ \mathcal{L}^{(\alpha, \eta)} \Phi_n^{(\alpha, \beta)}(x) \right\} \, d\Psi(x; \alpha + \eta, \beta).$$

Substituting from (2.12) we get

$$F_n(\alpha + \eta, \beta) \delta_{nn} = \left[ \frac{(q^\alpha; q)_\infty}{(q^{\alpha+\eta}; q)_\infty} \right]^2 \int_{-\infty}^\infty \sum_{k=0}^\infty \frac{(q^{\eta}; q)_k (q^{\eta}; q)_k}{(q; q)_k (q; q)_k} \cdot q^{(k+1)} \Phi_n^{(\alpha, \beta)}(x q^k) \Phi_n^{(\alpha, \beta)}(x q^k) \, d\Psi(x; \alpha + \eta, \beta).$$

Substituting for $d\Psi(x; \alpha + \eta, \beta)$ in (3.2) from (2.7) we get, after some simplification,

$$F_n(\alpha + \eta, \beta) \delta_{nn} = (q^\alpha; q)_\infty (q^\alpha; q)_\infty / (q^\beta; q)_\infty (q^{\alpha+\eta}; q)_\infty \cdot \sum_{r,s=0}^\infty \Phi_n^{(\alpha, \beta)}(q^r) \Phi_m^{(\alpha, \beta)}(q^s)$$

(3.3)

$$
\cdot (q^{\eta}; q)_r (q^{\eta}; q)_s / (q; q)_r (q; q)_s \cdot 3 \Phi_2(q^{-r}, q^{-s}, q^{\beta-a-\eta}; q^{-1-\eta-r}, q^{-1-\eta-s}, q^{2-\eta-a}).$$

Now using (2.6) and (3.3) we obtain
Combining (3.4) and (2.5) with the uniqueness of the orthogonal polynomials we get

\[
\Phi_n^{(\alpha, \beta)}(q^r) = \frac{(q^\alpha; q)_\infty (q^{\beta-\alpha}; q)_n (q^{\alpha+\eta}; q)_n}{(q^{\alpha+\eta}; q)_\infty (q^{\beta-\alpha-\eta}; q)_n (q^\alpha; q)_n} \frac{(q^\eta; q)_r}{(q^{-\alpha}; q)_r} q^{-m}
\]

(3.5)

\[
\sum_{s=0}^{\infty} \Phi_n^{(\alpha, \beta)}(q^s) q^{as} \frac{(q^\eta; q)_s}{(q; q)_s} \Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta}; q^{1-\eta-r}, q^{1-\eta-s}, q, q^2-\eta-\alpha) \).
\]

Clearly (3.5) is an integral equation satisfied by \( \Phi_n^{(\alpha, \beta)}(x) \). The kernel in (3.5) is not symmetric but can be symmetrized by rewriting (3.5) as

\[
q^{ar/2} \left\{ (q^{\beta-\alpha}; q)_r (q^\alpha; q)_\infty / (q; q) (q^{\beta}; q)_\infty \right\}^{1/2} \Phi_n^{(\alpha, \beta)}(q^r)
\]

\[
= \lambda_n(\alpha, \beta, \eta, q) \sum_{s=0}^{\infty} q^{as/2} \left\{ \frac{(q^{\beta-\alpha}; q)_s (q^\alpha; q)_\infty}{(q; q) (q^{\beta}; q)_\infty} \right\}^{1/2} \Phi_n^{(\alpha, \beta)}(q^s) K(q^s, q^r; \alpha, \beta, \eta, q).
\]

where \( K(q^s, q^r; \alpha, \beta, \eta, q) \) is the symmetric kernel

\[
K(q^s, q^r; \alpha, \beta, \eta, q) = q^{(r+s)a/2} (q^\eta; q)_r (q^\eta; q)_s
\]

(3.6)

\[
\cdot \left\{ (q^{\beta-\alpha}; q)_s (q^{\beta-\alpha}; q)_s (q; q) (q; q) \right\}^{-1/2} \Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta}; q^{1-\eta-r}, q^{1-\eta-s}, q, q^2-\eta-\alpha),
\]

and \( \lambda_n \) are the eigenvalues

\[
\lambda_n(\alpha, \beta, \eta; q) = \frac{(q^{\alpha+\eta}; q)_\infty (q^{\beta-\alpha}; q)_n q^{-m}}{(q^{\alpha+\eta+\eta}; q)_\infty (q^{\beta-\alpha-\eta}; q)_n} (n = 0, 1, 2, \ldots).
\]

The completeness of the system of orthogonal polynomials \( \{\Phi_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty} \) follows from Weierstrass' approximation theorem. Therefore, the eigenvalues given by (3.7) are all the eigenvalues of the integral equation

\[
f(q^r) = \lambda \sum_{s=0}^{\infty} K(q^s, q^r; \alpha, \beta, \eta; q) f(q^s), \quad r = 0, 1, 2, \ldots.
\]

The eigenvalues \( \lambda_n(\alpha, \beta; q) \) are positive and increasing to \( \infty \) for \( \alpha > 0 \).
\( \beta > \alpha + \eta \) and \( \eta > 0 \), and the symmetric kernel \( K(q^r, q^s; \alpha, \beta, \eta; q) \) belongs to \( L^2(d\mu) \) where

\[
\mu(x, y) = r(x)r(y),
\]

and

\[
dr(x) = \begin{cases} 
1 & \text{if } x = q^k, k = 0, 1, \ldots, \\
0 & \text{otherwise};
\end{cases}
\]

see §4. Now we are in a position to apply the extension of Mercer's theorem\(^2\) (see Tricomi [9, p. 125]) to the discrete integral equation (3.8) and derive the bilinear formula

\[
\sum_{n=0}^{\infty} f_n(q^r; \alpha, \beta) f_n(q^s; \alpha, \beta) \lambda_n(\alpha, \beta, \eta; q) = K(q^r, q^s; \alpha, \beta, \eta; q)
\]

with

\[
f_n(q^r; \alpha, \beta) = q^{\alpha r/2} \left[ \frac{\binom{q^r - \alpha; q}{q}_\infty (q^{\alpha}; q)_\infty}{(q; q)_\infty} \right]^{1/2} \Phi_n^{(\alpha, \beta)}(q^r).
\]

The reproducing kernel (3.6) can be iterated to give

\[
\sum_{n=0}^{\infty} \left[ \lambda_n(\alpha, \beta, \eta; q) \right]^{-1} f_n(q^r; \alpha, \beta) f_n(q^s; \alpha, \beta) = K^{(j)}(q^r, q^s; \alpha, \beta, \eta; q), \quad j = 1, 2, \ldots,
\]

and the iterated kernels \( K^{(j)} \) are defined inductively by

\[
K^{(0)}(q^r, q^s; \alpha, \beta, \eta; q) = K(q^r, q^s; \alpha, \beta, \eta; q),
\]

\[
K^{(j+1)}(q^r, q^s; \alpha, \beta, \eta; q) = \sum_{i=0}^{\infty} K^{(j)}(q^r, q^i; \alpha, \beta, \eta; q) K(q^{i}, q^s; \alpha, \beta, \eta; q).
\]

One can easily take special cases of the above bilinear formulas and derive similar formulas for the \( q \)-Laguerre and \( q \)-Legendre polynomials.

4. The square integrability of \( K(q^r, q^s; \alpha, \beta, \eta; q) \). In this section we shall prove that the kernel \( K(q^r, q^s; \alpha, \beta, \eta; q) \) of (3.6) belongs to \( L^2(d\mu) \), \( \mu \) being the measure defined by (3.4) and (3.10). Let \( \| K \|_\mu \) be the \( L^2(d\mu) \) norm of the above mentioned kernel. We now prove that \( \| K \|_\mu \) is finite.

From (3.5) we have, for \( \alpha > 0, \beta > \alpha + \eta, \eta > 0 \),

\[
\| K \|_\mu^2 = \sum_{r, s=0}^{\infty} \frac{(q^\eta; q)_\infty, (q^\eta; q)_\infty, q^{\alpha(r+s)}}{(q^\beta - \alpha; q)_\infty, (q^\beta - \alpha; q)_\infty, (q; q)_\infty, q^{\eta}} \cdot \left\{ 3 \Phi_2(q^{-r}, q^{-s}, q^{\beta - \alpha - \eta}; q^{1-\eta-r}, q^{1-\eta-s}; q, q^{2-\eta-\alpha}) \right\}^2.
\]

\(^2\)The extension is straightforward and we believe it is known.
Applying the Cauchy-Schwarz inequality on the \([\Phi_2]^2\) we get
\[
\|K\|_\mu^2 \leq \sum_{r,s=0}^{\infty} \frac{(q^n; q)_r^2 (q^n; q)_s^2 q^{\alpha(r+s)}}{(q^{\beta-a}; q)_r (q^{\beta-a}; q)_s (q; q)_r (q; q)_s} \cdot \sum_k (q^{-r}; q)_k^2 (q^{-s}; q)_k^2 (q^{\beta-a-n}; q)_k^2 \cdot \sum_k (q^{1-n-r}; q)_k^2 (q^{1-n-s}; q)_k^2 q^{2(2-n-a)k}.
\]

Interchanging summations and using the identity
\[
(q^{a-r}; q)_k = (-1)^k q^{k(a-r)+k(k-1)/2} \frac{(q^{1-a}; q)_r}{(q^{1-a}; q)_{r-k}},
\]
we get
\[
\|K\|_\mu^2 \leq \sum_{k=0}^{\infty} \frac{(q^{\beta-a-n}; q)_k^2 q^{2nk}}{(q^{\beta-a}; q)_k} \left[ \sum_r \frac{(r+k)q^{\alpha(r+k)} (q^{1+r}; q)_r (q^n; q)_r^2}{(q^{\beta-a+k}; q)_r (q; q)_r^2} \right]^2.
\]

The inside summation on the right-hand side of the above inequality behaves for large values of \(k\) as \((A + Bk)^2\) where \(A\) and \(B\) are constants. Hence the right-hand side is convergent (by the ratio test) under the stated conditions. This proves our assertion.

**Bibliography**


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