C(X) IS NOT NECESSARILY A RETRACT OF 2X

JACK T. GOODYKOONTZ, JR.

Abstract. Let X be a continuum and 2X(C(X)) denote the hyperspace of closed subsets (subcontinua) of X. An example is given of a continuum X such that C(X) is not a retract of 2X.

By a continuum X we mean a compact connected metric space. 2X denotes the hyperspace of closed subsets of X with the finite (Vietoris) topology. C(X) denotes the hyperspace of subcontinua of X with the relative topology from 2X. Since X is a continuum, each of 2X and C(X) is also a continuum (see [4]). If Y C X, then Y is a retract of X means that there exists a continuous function r: X → Y such that the restriction of r to Y is the identity on Y.

In this paper we obtain a partial answer to a question raised by Sam B. Nadler, Jr. In 1939 Wojdyslawski [7] proved that C(X) is an absolute retract if and only if X is locally connected. Hence C(X) is a retract of 2X when X is locally connected. In Theorem 3.6 of [5] Nadler shows that C(X) is always a continuous image of 2X. Following this result, in Question 3.7 of [5], Nadler asks “When is C(X) a retract of 2X?” Nadler discusses his question further in [6] and considers the possibility that C(X) is always a retract of 2X. Using an interesting approach with topological semilattices, Lawson [3] asserted that certain nonlocally connected continua admit a retraction from 2X onto C(X). However, there is an error in the proof of Proposition 4 in Lawson’s paper (his function βi is not a retraction from 2X onto C(X)). Consequently, there is currently no known example of a nonlocally connected continuum X such that C(X) is a retract of 2X. It is the purpose of this paper to present an example of a continuum X such that C(X) is not a retract of 2X. The example is a slightly modified version of a subcontinuum of Example 3 of [1].

If U1, . . . , Un are open sets in X, then N(U1, . . . , Un) = {A ∈ 2X | A ⊂ \bigcup_{i=1}^{n} U_i and for each i = 1, . . . , n, A ∩ U_i ≠ ∅}. The collection of sets N(U1, . . . , Un) forms a base for the finite topology on 2X (see [4]).

Example. This example is a subspace of E3 (Euclidean 3-space). If x, y ∈ E3 we let [x, y] denote the line segment in E3 joining x and y. For each
positive integer \( n \) let

\[ A_n = [(0, 1/2, 1/n), (0, 1, 0)] \]

and

\[ B_n = [(1, 0, 1/n), (1/n, 0, 1)] \cup [(1/n, 0, 1/n), (0, 1/2, 1/n)] \]

\[ \cup [(0, 1/2, 1/n), (-1/n, 0, 1/n)] \cup [(-1/n, 0, 1/n), (-1, 0, 1/n)]. \]

Let \( X \) be the closure of \( \bigcup_{n=1}^{\infty} (A_n \cup B_n) \) and let

\[ B = \lim_{n \to \infty} B_n = [(1, 0, 0), (-1, 0, 0)] \cup [(0, 0, 0), (0, 1/2, 0)]. \]

Suppose that \( r \) is a retraction from \( 2^X \) onto \( C(X) \). Let \( \mathcal{V} = N(V_1, \ldots, V_m) \cap C(X) \) be an open set such that \( B \in \mathcal{V} \) and such that each \( V_i \) has diameter \(< 1/2 \). Then \((0, 1, 0) \notin \mathcal{V} \). Since \( r(B) = B \) and \( r \) is continuous at \( B \), there exists an open set \( U = N(U_1, \ldots, U_k) \) such that \( B \in U \subset N(V_1, \ldots, V_m) \) and \( r(U) \subset \mathcal{V} \). Choose \( \delta > 0 \) such that, for

\[ D = [(1, 0, 0), (-1, 0, 0)] \cup [(0, 0, 0), (0, 1/2 - \delta, 0)], \]

\( D \in U \). Then there exists a sequence \( D_n \to D \) such that for each \( n \), \( D_n \subset B_n \) and \( D_n \in 2^X - C(X) \). Since \( D_n \to D \), \( r(D_n) \to r(D) = D \). Note that subcontinua of \( X \) which are sufficiently close to \( D \) must be contained in \( B \). Hence there exists a positive integer \( n_1 \) such that for each \( n > n_1 \), \( r(D_n) \subset B \). Since \( D_n \to D \in U \) and \( B_n \to B \in U \), there is a positive integer \( n_2 \) such that for each \( n > n_2 \), \( D_n \in U \) and \( B_n \in U \). It follows easily from the definition of the finite topology that if \( n > n_2 \) and \( E \in 2^X \) such that \( D_n \subset E \subset B_n \), then \( E \in U \). Let \( j = \max\{n_1, n_2\} \). By [2, Lemma 2.3] there is a segment \( \sigma: [0, 1] \to 2^X \) such that \( \sigma(0) = D_j \) and \( \sigma(1) = B_j \). Since \( j > n_2 \), it follows from [2, Definition 2.2] that \( \sigma([0, 1]) \subset \mathcal{V} \). Hence \( r(\sigma([0, 1])) \) is a subcontinuum of \( C(X) \) contained in \( \mathcal{V} \). Furthermore, \( r(D_j) \in r(\sigma([0, 1])) \) and \( B_j = r(B_j) \in r(\sigma([0, 1])) \). Let \( A = \bigcup r(\sigma([0, 1])) \). By [2, Lemma 1.2], \( A \) is a subcontinuum of \( X \). Since \( B_j \subset A \), \( r(D_j) \subset A \), and \( r(D_j) \subset B \), it is easy to see that \( (0, 1, 0) \in A \). However, since \( r(\sigma([0, 1])) \subset \mathcal{V} \), \( A \subset \bigcup \mathcal{V} \), and since \((0, 1, 0) \notin \bigcup \mathcal{V} \), \((0, 1, 0) \notin A \). We have a contradiction. Hence \( C(X) \) is not a retract of \( 2^X \).

REFERENCES


DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WEST VIRGINIA 26506