ON \(\mu\)-SPACES AND \(kR\)-SPACES

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Abstract. In this paper it is proved that when \(X\) is a \(kR\)-space then \(\mu X\) (the smallest subspace of \(\beta X\) containing \(X\) with the property that each of its bounded closed subsets is compact) also is a \(kR\)-space; an example is given of a \(kR\)-space \(X\) such that its Hewitt realcompactification, \(\nu X\), is not a \(kR\)-space. We show with an example that there is a non-\(kR\)-space \(X\) such that \(\nu X\) and \(\mu X\) are \(kR\)-spaces. Also we answer negatively a question posed by Buchwalter: Is \(\mu X\) the union of the closures in \(\nu X\) of the bounded subsets of \(X\)? Finally, without using the continuum hypothesis, we give an example of a locally compact space \(X\) of cardinality \(\aleph_1\) such that \(\nu X\) is not a \(k\)-space.

Introduction. The topological spaces used here will always be completely regular Hausdorff spaces. If \(X\) is a topological space we write \(C(X)\) for the ring of the continuous real-valued functions on \(X\), and \(\beta X\) (resp. \(\nu X\)) for the Stone-Čech compactification (resp. Hewitt realcompactification) of \(X\). A subset \(M\) of \(X\) is said to be bounded if \(g|_M\) is bounded for all \(g \in C(X)\). A space is said to be a \(\mu\)-space if every closed bounded subset is compact. Realcompact spaces (closed subspaces of a product of real lines) and \(P\)-spaces (spaces in which every \(G_\delta\) is open) are \(\mu\)-spaces. Write \(\mu X\) for the smallest subspace of \(\beta X\) that contains \(X\) and is a \(\mu\)-space. A real-valued function \(g\) on \(X\) is called \(kR\)-continuous if \(g|_K\) is continuous in \(K\) for all compact subsets \(K\) of \(X\). A space such that every \(kR\)-continuous function is continuous is called a \(k\)-space. The associated \(k\)-space of a space \(X\), denoted by \(kRX\), will be \(X\) provided with the coarsest topology for which every \(kR\)-continuous function on \(X\) is continuous. It is easy to see that \(kRX\) is a completely regular Hausdorff space.

Our work provides the solutions to the following questions:

1. If \(X\) is a \(kR\)-space, is \(\mu X\) a \(kR\)-space?
2. If \(\mathcal{B}\) is the family of all closed bounded subsets of \(X\), does the relation \(\mu X = \bigcup \{ B^{\nu X}: B \in \mathcal{B}\}\) hold?
3. If \(\nu X\) or \(\mu X\) is a \(kR\)-space, is \(X\) a \(kR\)-space?
4. If \(X\) is a \(kR\)-space, is \(\nu X\) a \(kR\)-space?
5. If \(X\) is a realcompact space, is \(kRX\) realcompact?\(^1\)

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\(^1\)This question has been proposed by M. Valdivia.
The answer to question (1) is affirmative. Moreover, if $X$ is a $k_R$-space, it is proven that $\bigcup \{ B^{vX}: B \in \beta \}$ is also, and that using this fact one has shown that $\mu X$ is a $k_R$-space. Question (2) has a negative answer; the example is given of a locally compact space $X$ such that $X \neq \bigcup \{ B^{vX}: B \in \beta \} \neq \mu X$. Using an example of Comfort [3], it is shown that the answer to question (3) is negative. Supposing continuum hypothesis (CH), an infinite class of $k_R$-spaces $X$ is constructed such that $vX$ is not a $k_R$-space. Thus question (4) has a negative answer. Further, it is shown that for the spaces $X$ constructed before, $k_R(vX)$ is not realcompact and therefore question (5) also has a negative answer.

Nachbin [12] and Shirota [16] give the following characterizations for $C_c(X)$ (the vector space $C(X)$ provided with the compact-open topology):

(NS$_1$) $X$ is realcompact if and only if $C_c(X)$ is bornological.
(NS$_2$) $X$ is a $\mu$-space if and only if $C_c(X)$ is barrelled.

These characterizations provide examples of nonbornological barrelled spaces, answering a question posed by Bourbaki [1]. Warner [18] characterizes the $k_R$-spaces as those spaces $X$ for which $C_c(X)$ is complete (W).

Question 5 is related to the following problem posed by Dieudonné [4]: If $E$ is a bornological space, is the completion of $E$ a bornological space? T. Kōmura and Y. Kōmura [7], supposing (CH), give an example of a bornological space whose completion is not bornological, using sequence spaces of Köthe [8]. Since De Wilde and Schmets have proven [19] that $X$ is realcompact if and only if $C_c(X)$ is an ultrabornological space (inductive limit of Banach spaces), the examples of realcompact spaces $X$ such that $k_RX$ is not realcompact provide ultrabornological spaces $C_c(X)$ such that the completion is not a bornological space. Thus, our solution to question (5) is another solution to the problem proposed by Dieudonné, in the context of spaces $C_c(X)$.

A space is said to be a $k$-space if each of its subsets which has closed intersection with each compact subset is itself closed. Evidently each $k$-space is a $k_R$-space, but Pták presents [15] an example (credited to Katětov) which shows that the converse implication can fail. Another (completely regular Hausdorff) example has been discovered by Noble [14]. In [3], Comfort gives an example of a locally compact space $X$ whose cardinality is $\kappa_2$ such that $vX$ is not a $k$-space and asks himself if there exists a space of cardinality $\kappa_1$ with the same properties as the former one. In [13] Negrepontis, supposing (CH), gives an example of a space with these properties. We give a different example from the former one, this one without (CH).

**Question 1.** If $X$ is a topological space, let $\mu X$ be the intersection of all subspaces of $\beta X$ which contain $X$ and are $\mu$-spaces. Thus, $\mu X$ is a $\mu$-space such that $X \subset \mu X \subset vX$. It can be easily shown that $X$ is a $\mu$-space if and only if $X = \mu X$, and that $X$ is compact if and only if it is a pseudocompact $\mu$-space. The space $\mu X$ is unique in the following sense: If $T$ is a $\mu$-space which contains $X$ as a dense subspace and every continuous mapping $\tau$ from
$X$ into any $\mu$-space $Y$ has a continuous extension $\tilde{\tau}$ from $T$ into $Y$, then there exists a homeomorphism of $\mu X$ onto $T$ that leaves $X$ pointwise fixed. Indeed, let $\phi$ be a continuous mapping from $X$ into any $\mu$-space $Y$; then $\phi$ has a Stone extension $\hat{\phi}$ to the whole $\beta X$ into $\beta Y$. If we prove that $E = \hat{\phi}^{-1}(Y)$ is a $\mu$-space, then the restriction of $\hat{\phi}$ to $\mu X$ is a continuous extension of $\phi$ because $\mu X \subset E$. If $A$ is a bounded subset of $E$, then $B = \hat{\phi}(A)$ is a compact subset of $Y$ and since $\hat{\phi}^{-1}(B)$ is compact, the subset $A$ is relatively compact in $E$. Therefore every continuous mapping from $X$ into any $\mu$-space $Y$ has a continuous extension from $\mu X$ into $Y$, and according to [5, 0.12] the proof is complete.

**Proposition 1.** Let $X$ be a topological space, let $\mathcal{B}$ be the family of all bounded subsets of $X$, and let $E(X) = \bigcup \{ B^{\infty} : B \in \mathcal{B} \}$. If $X$ is a $k_R$-space, then so is $E(X)$.

**Proof.** If $f$ is a $k_R$-continuous function in $E(X)$, and $g$ is the restriction of $f$ to $X$, then $g \in C(X)$. If $h$ is the continuous extension of $g$ to $E(X)$ and $x \in E(X) \sim X$, then $x \in A^{\infty}$, $A \in \mathcal{B}$, since $K = A^{\infty} = A^{\infty}(x)$ is compact, $f(x) = h(x)$ and $f = h$.

If $\alpha$ is an ordinal, we write $W(\alpha)$ for the set of all ordinals less than $\alpha$. If $M$ is a set, we denote the cardinal of $M$ by $|M|$. If $\alpha$ is a cardinal larger than $2^{||\omega\|}$, let $\omega_\alpha$ be the first ordinal whose cardinal is $\omega_\alpha$. We define inductively $\{ B_\alpha : \sigma \in W(\omega_\alpha) \}$, where $B_1 = E(X)$ and $B_\alpha = E(\bigcup \{ B_\delta : \delta < \sigma \})$. Let us suppose that $B_\delta$ is a $k_R$-space for every $\delta < \sigma$, $\sigma \in W(\omega_\alpha)$ and we shall prove that $B_\alpha$ is a $k_R$-space. By Proposition 1 it suffices to prove that $W = \bigcup \{ B_\delta : \delta < \sigma \}$ is a $k_R$-space. If $f$ is a $k_R$-continuous function in $W$, let $g$ be the continuous extension to $W$ of the restriction of $f$ to $X$. If $x \in B_{\delta_\alpha}$, $\delta_\alpha < \sigma$, since $g$ and $f$ are continuous in $B_{\delta_\alpha}$ and coincide over $X$, it follows that $f(x) = g(x)$ and therefore, $f = g$. Thus, $f$ is continuous in $W$ and so $W$ is a $k_R$-space. Let us now suppose that $\mu X \neq B_\sigma$ for every $\sigma \in W(\omega_\alpha)$, and we choose $\sigma_1 \in W(\omega_\alpha)$ such that $|\sigma_1| > |\mu X|$ (we write $|\sigma_1|$ for the cardinal number of $\sigma_1$). For every $\gamma < \sigma_1$ we choose a point $x_\gamma \in B_\gamma \sim \bigcup \{ B_\delta : \delta < \gamma \}$. Therefore $|B_{\sigma_1} \sim X| > |\sigma_1|$ because $|\sigma_1| = |\{ x_\gamma, \gamma \in W(\sigma_1) \}|$ and $x_\gamma \in B_{\sigma_1} \sim X$ for all $\gamma < \sigma_1$. On the other hand, the relation $|\sigma_1| > |\mu X| > |\mu X \sim X| > |B_{\sigma_1} \sim X|$ holds, which is a contradiction, so there exists $\alpha_0 \in W(\omega_\alpha)$ such that $B_{\alpha_0} = \mu X$.

**Question 2.** First, we shall give an example of a locally compact space $X$ such that $|X| = \kappa$, and that $\nu X$ is not a $k$-space. If $\alpha$ is an ordinal, we write $\alpha + 1$ for the ordinal which follows it and $\omega_0$ (resp. $\omega_1$) for the first infinite (resp. uncountable) ordinal. Let $Y$ be the product space $W(\omega_1 + 1) \times \nu X$ such that $|X| = \kappa$, and that $\nu X$ is not a $k$-space. If $\alpha$ is a limit ordinal ($\alpha < \omega_1$), let $\{ \beta_n \}_{n=1}^{\infty}$ be a strictly increasing sequence in $W(\alpha)$ which converges to $\alpha$. If $\gamma_n = \beta_n + 1$, $\alpha_n = \gamma_n + 1$, $n = 1, 2, \ldots$, it follows that $\{ \alpha_n \}_{n=1}^{\infty}$ is a strictly increasing
sequence which converges to $\alpha$, $\alpha_n$ being an isolated point of $W(\omega_1)$. If $p$ is a positive integer we write $A_{p,\alpha} = \{(\delta, p) : \alpha_p < \delta < \alpha\}$ and $U_{n,\alpha} = \{(\alpha, \omega_1) \cup \bigcup_{p=n}^{\infty} A_{p,\alpha}\}$. If $\alpha$ is a limit ordinal of $W(\omega_1)$ let $f_\alpha$ be the function defined on $Y$ as $f_\alpha(\alpha, \omega_1) = 0$, $f_\alpha(A_{n,\alpha}) = \{1/n\}$, $n = 1, 2, \ldots$, and, otherwise, as $1$. Let $\mathcal{F}$ be the weak topology on $Y$ associated to $C(Y)$ and to the family of functions $\{f_\alpha, \alpha \text{ a limit ordinal of } W(\omega_1)\}$. Then $(Y, \mathcal{F})$ is a nonpseudocompact completely regular space, because $\{((\gamma_n, n))_{n=1}^{\infty}\}$ is a copy of $N$ (discrete space of positive integers), which is $C$-embedded in $Y$. For the topology $\mathcal{F}$ a basis of the neighborhoods of this point in the product topology. The same is true for $(\omega_1, \omega_0)$ and $(\eta, \omega_0)$ when $\eta$ is a nonlimit ordinal of $W(\omega_1)$. If $\eta$ is a limit ordinal of $W(\omega_1)$, a basis of the neighborhoods of $(\eta, \omega_0)$ is the family $\{U_{n,\eta} : n = 1, 2, \ldots\}$. If $X = Y \sim \{(\omega_1, \omega_0)\}$, let us see that $X$ is locally compact. If $\eta$ is a limit ordinal of $W(\omega_1)$ and $\{V_i : i \in L\}$ is an open cover of $U_{i,\eta}$ there exists $n_0 \in N$, $i_0 \in L$ such that $U_{n_0,\eta} \subset V_{i_0}$. If $\{V_j : 1 < j < K\}$ is a finite subcover of the compact set $\bigcup \{A_{p,\alpha} : 1 < p < n_0\}$, then $\{V_j : 0 < j < K\}$ is a finite subcover of $U_{i_0,\eta}$. Now we shall prove that $X$ is $C$-embedded in $Y$. If $f \in C(X)$, there exists $\gamma \in W(\omega_1)$ such that if $\beta > \gamma$, then $f(\beta, n) = f(\omega_1, n)$, $n = 1, 2, \ldots$. By continuity, it results that $f(\beta, \omega_0) = f(\gamma, \omega_0)$ if $\beta > \gamma$, $\beta < \omega_1$. If $\hat{f}$ is the function that coincides on $A$ with $f$ and $\hat{f}(\omega_1, \omega_0) = f(\gamma, \omega_0)$ it follows that $\hat{f}$ is a continuous extension of $f$. If we prove that $Y$ is realcompact, we shall have $Y = \nu X$. Let us suppose that $M$ is a free real maximal ideal of $C(Y)$. Then $M \neq \{f \in C(Y) : f(\omega_1, \omega_0) = 0\}$ and if $Z(M) = \{Z(g) : g \in M\}$ there will be $Z^1 \in Z(M)$, $\sigma_0 \in W(\omega_1)$ such that $Z^1 \cap \{(\beta, \omega_0) : \sigma_0 + 1 < \beta \leq \omega_1\} = \emptyset$. Since $H_\alpha = \{(\alpha, n) : 1 < \alpha \leq \omega_1\}$ is a compact zero-set, then $H_\alpha \notin Z(M)$. Thus there exists $Z_\alpha \in Z(M)$ such that $Z_\alpha \cap H_\alpha = \emptyset$, $n = 1, 2, \ldots$. Since $M$ is real, if $Z^2 = \bigcap_{n=1}^{\infty} Z_\alpha$, then $Z^2 \subset Z(M)$ and, therefore, $Z = Z^1 \cap Z^2 \subset Z(M)$, $Z \subset \{(\beta, \omega_0) : 1 < \beta \leq \sigma_0\}$, and so $Z = \{((\gamma_K, \omega_0))_{K=1}^{\infty} : 1 < \gamma_1 < \gamma_2 < \cdots < \sigma_0\}$. Because $\{(\gamma_K, \omega_0)\}$ is a zero-set in $Y$, $K = 1, 2, \ldots$, and since $M$ is real, there exists $K_0 \in N$ such that $\{(\gamma_{K_0}, \omega_0)\} \subset Z(M)$ and, therefore, $M$ is not free. This contradiction shows us that no free real maximal ideals in $C(Y)$ exist, and that, therefore, $Y$ is realcompact. As the set $\{(\sigma, \omega_0) : 1 < \sigma \leq \omega_1\}$ meets the compact subsets of $Y$ in closed sets but is not closed, it results that $Y$ is not a $k$-space.

We are now going to resolve negatively question (2) with an example. Let $T$ be the subspace $Y \sim \{(\omega_1, n) : 1 < n \leq \omega_0\}$, which is locally compact, and we shall prove that $T \neq \bigcup \{B^{\nu T} : B \in \mathbb{R}\}$. Being the family of all closed bounded sets of $T$. Since $T$ is $C$-embedded in $Y$ it follows that $Y = \nu T = \nu X$ and so $\mu T \subset \mu X$. The equality $X = \bigcup \{B^{\nu T} : B \in \mathbb{R}\}$ is a direct consequence of the following lemmas.

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2We say that $I$ is an ideal of $C(X)$ if it is a subring of $C(X)$ and if $f \in I$, $g \in C(X)$ implies $gf \in I$. An ideal $I$ of $C(X)$ is said to be free when $\bigcap \{Z(f) : f \in I\} = \emptyset$. A space $X$ is realcompact if for every free maximal ideal $I$ of $C(X)$, the residue class ring $C(X)/I$ is not isomorphic with the ring of the real numbers.
LEMMA 1. If $A \subset T$ and $\{\sigma_K\}_{K=1}^\infty$ is a strictly increasing sequence in $W(\omega_1)$ such that $(\sigma_K, \omega_0) \in A, K = 1, 2, \ldots$, then $A$ is not bounded in $T$.

PROOF. If $\eta \in W(\omega_1)$ is not a limit ordinal we write $B(K, n_K) = \{\eta, n\)$, $n_K < n < \omega_0$ and if it is a limit ordinal then $B(K, n_K) = U_{n_-, \gamma}$ where $\gamma = \sup K \sigma_K$ and for $1 \leq j \leq K - 1$, $B(j, n_j) \cap B(K, n_K) = \emptyset$, $n_1 < n_2 < \cdots < n_K$. The function whose value is $K$ in $B(K, n_K), K = 1, 2, \ldots$, and vanishes otherwise, is continuous in $T$ and nonbounded in $A$.

LEMMA 2. If $A$ is a closed set in $T$ and $(\omega_1, \omega_0) \in A'$, then $A$ is not bounded in $T$.

PROOF. Since $(\omega_1, \omega_0) \in A'$ it is possible to choose a sequence $\{(\gamma_n, n_n)\}_{n=1}^\infty$ in $A$ such that $n_{K+1}^1 < n_{K+1}^{2}, \gamma_{K+1}^1 < \gamma_{K+1}^2, K = 1, 2, \ldots$. With the product topology this sequence converges to $(\sigma_1, \omega_0)$ where $\sigma_1 = \sup K \gamma_K^1$. If this sequence does not converge with the topology $\mathcal{T}$ to $(\sigma_1, \omega_0)$, then it is a discrete closed set $C$-embedded in $T$ contained in $A$. Therefore $A$ is not bounded in $T$ and the lemma is proved. If the sequence converges to $(\sigma_1, \omega_0)$ in the topology $\mathcal{T}$, we can consider a sequence $\{(\gamma_n^2, n_n^2)\}_{n=1}^\infty$ in $A$ satisfying $n_{K+1}^2 < n_{K+1}^{2}, \gamma_{K+1}^2 > \gamma_{K+1}^1 > \sigma_1, K = 1, 2, \ldots$, and we shall proceed as before. If for $p = 1, 2, \ldots$, the sequence $\{(\gamma_n^p, n_n^p)\}_{n=1}^\infty$ converges to $(\sigma_p, \omega_0)$ with the topology $\mathcal{T}$, then $(\sigma_p, \omega_0) \in A, \sigma_p < \sigma_{p+1}, p = 1, 2, \ldots$, and therefore, from Lemma 1, the set $A$ is not bounded in $T$.

Returning to our example, from Lemma 2 we deduce that $X = \bigcup \{B_n^\mu, B \in \mathfrak{B}\} \subset \mathfrak{U}$, so that $\mu X \subset \mu T$ and consequently $\mu X = \mu T$. Thus, $X \neq \mu X$ since the set $\{(\omega_1, n): 1 \leq n < \omega_0\}$ is closed and bounded in $X$ and noncompact.

Note. Since $Y = \mu X = \nu X$, by Proposition 1 it results that $Y$ is a $k_R$-space. In [17] it is proved that $k Y^3$ is not a regular space.

This example provides a solution to problem 2 of Buchwalter [2].

Question 3. Now we give an example of a topological space $X$ such that $\mu X = \nu X$ is a $k_R$-space and $X$ is not a $k_R$-space. Comfort [3] gives an example of a pseudocompact space $X$ whose cardinality is $c$ such that $N \subset X \subset \beta N$. Since every infinite closed set in $\beta N \sim N$ has cardinality $2^c$ [5, 9.12], it follows that the compact subsets of $X$ are finite. But $X$ is not discrete, so that it is not a $k_R$-space and $\mu X = \nu X = \beta X$.

Question 4. Firstly, let us note that every quotient space of a $k_R$-space is a $k_R$-space. This result is intimately connected with the fact that $k_R$ is a coreflector (see [6] or [10]). Thus, if a topological product is a $k_R$-space then each factor space also is.

A subset of a topological space $X$ is said to be $k$-closed if it intersects every

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3If $X$ is a topological space, the associated $k$-space to $X$, denoted by $kX$, will be $X$ provided with the topology for which a set is closed if and only if it intersects every compact set of $X$ in a closed set.
compact subset of $X$ in a closed set of $X$. Therefore, a $k$-space is a space in which every $k$-closed subset is closed.

**Proposition 2.** If $X$ is a $\mu$-space and $vX \sim X$ is $k$-closed in $vX$, then $vX$ is not a $k_R$-space.

**Proof.** Since $X$ is a $\mu$-space $C$-embedded in $vX$, then $X$ is $k$-closed in $vX$. Moreover $vX \sim X$ is $k$-closed in $vX$. Therefore, the function whose value is 0 on $X$ and 1 on $vX \sim X$ is $k_R$-continuous but is not continuous in $vX$.

**Theorem 2.** Let $\{U_i\}_{i \in I}$ be a family of clopen pairwise disjoint sets in $\beta N \sim N$ and let $X = N \cup \{\bigcup_{i \in I} U_i\}$. If $|vX \sim X| < 2^c$ and the points of $vX \sim X$ are not adherent to any countable subset in $X \sim N$, then $X$ is a locally compact $\mu$-space and $vX$ is not a $k_R$-space.

**Proof.** As $\beta X = \beta N$ and $X$ is open in $\beta X$, it results that $X$ is locally compact. Let us see that $X$ is a $\mu$-space. Suppose that there is a noncompact closed bounded subset $A$ in $X$. Since $vX$ is a $\mu$-space it follows that $K = A^{vX}$ is compact. If $A$ intersects infinitely many $U_i$ we can choose a sequence $x_n \in A \cap U_i$, $n = 1, 2, \ldots$, with $i_n \neq i_m$ if $n \neq m$, such that $(x_n)^\infty_{n=1}$ has no adherent points in $N$ and $X \sim N$ because $U_i$ is open in $X \sim N$ for every $i \in I$ and $U_i \cap U_j = \emptyset$, $i \neq j$. Further, this sequence has no adherent points in $vX \sim X$ by hypothesis, and therefore $K$ is not compact. Thus, there is a finite sequence $i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n$ such that $A \cap U_i = \emptyset$, $1 \leq j \leq n$, then $A \cap N \cap W$ is a nonempty bounded subset in $X$ and $K_0 = A \cap N \cap W^{vX}$ is compact. So $K_0 \cap U_i = \emptyset$ for every $i \in I$ and, therefore, $|K_0| < 2^c$, which is a contradiction since the infinite compact subsets of $\beta N$ have cardinality equal to $2^c$. Thus every bounded closed subset of $X$ is compact. On the other hand, $vX \sim X$ is closed in $vX$ and from Proposition 2 we have that $vX$ is not a $k_R$-space.

A point $x \in X$ is a $P$-point in $X$ if $Z(f)$ is a neighborhood of $x$ for all $f \in C(X)$ such that $f(x) = 0$. Then, $X$ is a $P$-space if and only if every point is a $P$-point in $X$.

Let us now look at an example of a space satisfying the hypothesis of Theorem 2. Assume (CH). According to Rudin [5, 6V] there is a $P$-point $p$ in $\beta N \sim N$. Let $\{G_\alpha\}_{\alpha < \omega_1}$ be a basis of clopen neighborhoods for $p$, such that $G_\alpha \subset G_\beta$, $G_\alpha \neq G_\beta$ for all $\beta < \alpha$. We define inductively a family of nonempty clopen sets $\{V_\alpha\}_{\alpha < \omega_1}$ in $\beta N \sim N$ such that $V_\alpha \subset G_\alpha \sim G_{\alpha+1}$ for all $\alpha < \omega_1$. If $X = N \cup \{\bigcup_{\alpha < \omega_1} V_\alpha\}$, according to Negrepontis [13], $vX = X \cup \{p\}$ and since $p$ is a $P$-point in $\beta N \sim N$, is not adherent to any sequence in $X \sim N$. From Theorem 2 we deduce that $X$ is a $k_R$-space and that $vX$ is not, with which question (4) is negatively resolved.

From this example we shall give an infinite class of $k_R$-spaces $Z$ for which $vZ$ is not a $k_R$-space. By a $\{0,1\}$-valued measure on a set $F$, we mean a countably additive function defined on the family of all subsets of $F$ and
assuming only the values 0 or 1. We call a cardinal $m$ measurable if a set $F$ of cardinal $m$ admits a $\{0,1\}$-valued measure $\sigma$ such that $\sigma(F) = 1$ and $\sigma(\{x\}) = 0$ for every $x \in F$. A discrete space is realcompact if and only if its cardinal is nonmeasurable [5, T.12.2].

**Theorem 3.** Let $X$ be a locally compact space such that $vX$ is not a $k_R$-space. Let $Y$ be a pseudocompact $k$-space and suppose that $X \times Y$ has nonmeasurable cardinal. Then $\mu(X \times Y)$ is a $k_R$-space and $v(X \times Y)$ is not.

**Proof.** The locally compact spaces are characterized by the property that the products with $k$-spaces are $k$-spaces [11, T.3.1]. Therefore $X \times Y$ is a $k$-space and from Theorem 1, $\mu(X \times Y)$ is a $k_R$-space. On the other hand, $v(X \times Y) = vX \times vY$ [3, T.2.4], and since $vX$ is not a $k_R$-space, it follows that $v(X \times Y)$ is not a $k_R$-space.

From (W), (NS), and (NS2), we now obtain the following corollary.

**Corollary.** Assuming (CH), there exist infinite spaces $Z$ for which $C_\ell(Z)$ is a complete, barrelled, and nonbornological space.

Note. If $X$ is a discrete space of measurable cardinal then $vX$ is a nondiscrete $P$-space and, therefore, $vX$ is not a $k_R$-space.

**Question 5.** The main result which we shall now prove is that for $k_R$-spaces $X$ the condition that $vX$ is not a $k_R$-space is equivalent to the fact that $k_R(vX)$ is not realcompact. In [9] the following theorem is proved:

**Theorem A.** Let $(X, \mathcal{T})$ be a completely regular Hausdorff space and let $Y$ be a subset in $\beta X$ which strictly contains $X$. Let $\mathcal{T}_0$ be a topology on $Y$ strictly finer than the induced topology by $\beta X$ such that the restriction of $\mathcal{T}_0$ to $X$ coincides with $\mathcal{T}$ and that $X$ is a dense subset in $Y$ for $\mathcal{T}_0$. Then $(Y, \mathcal{T}_0)$ is not a completely regular space.

If $X$ is a topological space provided with the topology $\mathcal{T}$ and $M$ is a subset of $X$, we denote by $M[\mathcal{T}]$ the set $M$ provided with the topology induced by $\mathcal{T}$.

**Theorem 4.** If $M$ is a $k_R$-space, then the following conditions are equivalent:

(a) $vM$ is not a $k_R$-space.

(b) The associated $k_R$-space to $vM$ is not realcompact.

**Proof.** That (b) implies (a) is trivial. We are going to prove that (a) implies (b). Write $\mathcal{T}$ (resp. $\mathcal{T}_0$) for the topology of $X = vM$ (resp. $k_R(X)$). Suppose that $X$ is not a $k_R$-space. Since $X \neq k_R X$ we have that $\mathcal{T}_0$ is strictly finer than $\mathcal{T}$. Thus, since $M[\mathcal{T}]$ is a $k_R$-space, it follows that both topologies coincide on $M$. According to Theorem A, $M$ is not dense in $k_R X$. Suppose that $k_R X$ is realcompact and let $H = \overline{M[\mathcal{T}]}$. Then $X \neq H$ and $H[\mathcal{T}_0]$ is realcompact. Let us now see that $C_\emptyset(H) \subset C_\emptyset(H)$, where $C_\emptyset(H)$ (resp. $C_\emptyset(H)$) is the ring of all continuous real-valued functions on $H[\mathcal{T}]$ (resp. $H[\mathcal{T}_0]$). If $f \in C_\emptyset(H)$ and $g = f|_M$ then $g \in C(M)$ and there is an extension $\hat{g} \in C_\emptyset(H)$ of $g$ to $H$, because $M$ is $C$-embedded in $X$. Thus, $\hat{g} \in C_\emptyset(H)$ and $\hat{g}|_M = f|_M$ and
therefore \( f = \hat{g} \) and \( f \in C_q(H) \). From here it results that \( C_q(H) = C_q(H) \) and that therefore \( H[\mathcal{F}] \) is realcompact, which is impossible, since \( X = vM \) and \( X \neq H \). Then \( k_RX \) is not realcompact.

As a consequence of Theorem 3 we have the following

**Corollary.** Let \( X \) be a locally compact space such that \( vX \) is not a \( k_R \)-space and let \( Y \) be a pseudocompact \( k \)-space. If \( X \times Y \) has nonmeasurable cardinal, then \( k_R(v(X \times Y)) \) is not realcompact.

**Note.** If \( X \) is a discrete space of measurable cardinal we know that \( vX \) is not a \( k_R \)-space and, according to Theorem 4, \( k_R(v(X \times Y)) \) is not realcompact.

I am informed that Question 2 was also proven by R. Haydon with a different example.

**References**


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