TRANSFORMATIONS INTO BAIRE 1 FUNCTIONS

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Abstract. A measurable \( f \) from \( I = [0, 1] \) to \( \mathbb{R} \) is equivalent to a Baire 2 function but may not be equivalent to any Baire 1 function. Gorman has obtained the following interesting contrasting facts. If \( f \) assumes finitely many values there is a homeomorphism \( h \) of \( I \) such that \( f \circ h \) is equivalent to a Baire 1 function, but there is a measurable \( f \) which assumes countably many values which does not have this property. However, the example of Gorman is such that for some homeomorphisms \( h \) the function \( f \circ h \) is not measurable. It is shown here that if \( f \circ h \) is measurable, for every homeomorphism \( h \), then there is an \( h \) for which \( f \circ h \) is equivalent to a Baire 1 function.

1. A measurable \( f \) from \( I = [0, 1] \) to \( \mathbb{R} \) is equivalent to a Baire 2 function but may not be equivalent to any Baire 1 function. Gorman [1] has obtained the following interesting contrasting facts. If \( f \) assumes finitely many values there is a homeomorphism \( h \) of \( I \) such that \( f \circ h \) is equivalent to a Baire 1 function, but there is a measurable \( f \) which assumes countably many values which does not have this property. However, the example of Gorman is such that for some homeomorphisms \( h \) the function \( f \circ h \) is not measurable.

A function \( f \) is said to be absolutely measurable if for every homeomorphism \( h \) of \( I \) the function \( f \circ h \) is measurable. This is tantamount to saying that \( f \) is measurable with respect to every Lebesgue-Stieltjes measure derived from a strictly increasing continuous distribution function. We prove the following result.

**Theorem.** If \( f : I \to \mathbb{R} \) is absolutely measurable there is a homeomorphism \( h \) of \( I \) such that \( f \circ h \) is equivalent to a Baire 1 function.

2. We give some preliminary definitions and lemmas. A set \( E \subset I \) is of absolute measure zero if for every homeomorphism \( h \) the set \( h(E) \) is of measure zero. A point \( x \in I \) is a c-point of a set \( E \) if for every neighborhood \( N \) of \( x \) the set \( N \cap E \) has cardinality \( c \) and \( x \) is a perfect c-point of \( E \) if for every neighborhood \( N \) of \( x \) the set \( N \cap E \) contains a nonempty perfect set. \( E \) is c-dense (perfectly dense) in a set \( D \) if every point of \( D \) is a c-point (perfect c-point) of \( E \). Gorman [2] has obtained the following lemma.

**Lemma 1.** If \( E \subset I \) is of the first category there is a homeomorphism \( h \) of \( I \)

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such that \( h(E) \) is of measure zero. If \( E \subseteq I \) is an \( F_\alpha \) set and is \( c \)-dense in \( I \) there is a homeomorphism \( h \) of \( I \) such that \( h(E) \) has measure one.

For every \( E \subseteq I \) let \( P(E) \) be the set of points in \( I \) which are perfect \( c \)-points of \( E \).

**Lemma 2.** If \( E \subseteq I \) is absolutely measurable the set \( E \setminus P(E) \) is of absolute measure 0.

**Proof.** Since \( P(E) \) is closed, \( E \setminus P(E) \) is absolutely measurable. Since \( E \setminus P(E) \) has no perfect \( c \)-points neither does \( h(E \setminus P(E)) \) for any homeomorphism \( h \) of \( I \). So \( h(E \setminus P(E)) \) has measure 0.

Let \( E \) be absolutely measurable and let \( G(E) = P(E)^0 \). Partition \( E \) into 3 sets, \( E^* = E \cap G(E) \), \( E^{**} = E \cap [P(E) \setminus G(E)] \), and \( E^{***} = E \setminus P(E) \).

**Lemma 3.** \( E^* \) is perfectly dense in \( G(E) \), \( E^{**} \) is nowhere dense, and \( E^{***} \) is of absolute measure 0.

**Proof.** By Lemma 2.

**Lemma 4.** Let \( J = \bigcup_1^\infty E_n \), where \( J \subseteq I \) is an open interval and each is absolutely measurable. There is an \( n \) such that \( G(E_n) \) is nonempty.

**Proof.** Otherwise \( J = (\bigcup_1^\infty E_n^{**}) \cup (\bigcup_1^\infty E_n^{***}) \). The set \( \bigcup_1^\infty E_n^{**} \) is of the first category so that by Lemma 1 there is a homeomorphism \( h \) such that \( h(\bigcup E_n^{**}) \) is of measure 0. But \( \bigcup E_n^{***} \) is of absolute measure 0 so that \( h(\bigcup E_n^{**}) \) is also of measure 0.

3. We now prove the main lemma.

**Lemma 5.** If \( f: I \to R \) is absolutely measurable and takes only countably many values, there is a homeomorphism \( h \) of \( I \) such that \( f \circ h \) is equivalent to a Baire 1 function.

**Proof.** Let \( \{a_n\} \) be the sequence of values assumed by \( f \) and let \( E_n = f^{-1}(a_n) \), \( n = 1, 2, \ldots \). The sets \( E_n \) are absolutely measurable and their union is \( I \). Denote the open components of \( P(E_1)^0 \) by \( I_{11}, I_{12}, \ldots \) and the open components of \( I \setminus P(E_1) \) by \( J_{11}, J_{12}, \ldots \), and for each \( n = 2, 3, \ldots \) the open components of \( P(E_n)^0 \cap (\bigcup_m J_{n-1,m}) \) by \( I_{n1}, I_{n2}, \ldots \) and the open components of \( (\bigcup_m J_{n-1,m}) \setminus P(E_n) \) by \( J_{n1}, J_{n2}, \ldots \). It follows by Lemma 4 that the union \( G \) of all the \( I_{nm} \) is a dense open subset of \( I \). By Lemma 1, there is a homeomorphism \( h_1 \) of \( I \) such that \( h_1^{-1}(G) \) has measure one. Now \( E_n \cap I_{nm} \) contains an \( F_\alpha \) set which is \( c \) dense in \( I_{nm} \). So by Lemma 1, the homeomorphism \( h_1 \) may be modified on each \( h_1^{-1}(I_{nm}) \) to a homeomorphism \( h \) so that \( h^{-1}(E_n \cap I_{nm}) \) is of full measure in \( h^{-1}(I_{nm}) \). Define

\[
g(x) = \begin{cases} 
a_n & \text{if } x \in h^{-1}(I_{nm}) \text{ for some } n, m, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( g \) is Baire 1 and \( f \circ h \) is equal almost everywhere to \( g \).
4. The following lemma serves as a bridge for carrying our result from countable valued functions to arbitrary absolutely measurable functions.

**Lemma 6.** If \( f: I \rightarrow R \) is absolutely measurable, and \( P \subseteq I \) is perfect there is a perfect \( Q \subseteq P \) such that \( f|Q \) is continuous.

**Proof.** There is a homeomorphism \( h \) of \( I \) such that \( h^{-1}(P) \) has positive measure. By Lusin's theorem \( f \circ h \) is continuous on a perfect subset \( R \) of \( h^{-1}(P) \) of positive measure. Set \( Q = h(R) \).

We also need the following two lemmas.

**Lemma 7.** Let \( f: I \rightarrow R \). There is a homeomorphism \( h \) of \( I \) such that \( f \circ h \) equals a Baire 1 function almost everywhere if and only if there is a \( c \)-dense \( F_\sigma \) set \( E \) and a Baire 1 function \( g \) such that \( f(x) = g(x) \) for every \( x \in E \).

**Proof.** This follows from Lemma 1 and the fact that a set of full measure in \( I \) contains an \( F_\sigma \) set which is a \( c \)-dense in \( I \).

**Lemma 8.** If \( E \subseteq I \) is of type \( F_\sigma \) and is \( c \)-dense in an open set \( H \) and \( S \) is the union of a nowhere dense set and a set of absolute measure 0, then \( E \setminus S \) contains an \( F_\sigma \) set \( F \) which is \( c \)-dense in \( H \).

**Proof.** There is a homeomorphism \( h \) such that \( h(H) \) and \( h(E) \) have the same measure. But \( h(S) \) is the union of a nowhere dense set and a set of measure 0. Then \( h(E \setminus S) \) contains an \( F_\sigma \) set \( K \) which is \( c \)-dense in \( h(H) \), and \( h^{-1}(K) \subseteq E \setminus S \) is \( c \)-dense in \( H \).

5. We are now ready to prove the theorem.

**Theorem 1.** Every absolutely measurable function on \( I = [0, 1] \) can be transformed by a homeomorphic change of variable into a function which is equal almost everywhere to a Baire 1 function.

**Proof.** Let \( f \) be absolutely measurable. By Lemma 7, it suffices to show that there is a \( g \) in Baire 1 such that \( f(x) = g(x) \) on a set containing a \( c \)-dense \( F_\sigma \). Let \( \{f_k\} \) take on only values of the form \( m/2^{k-1} \) such that

\[
(i) \quad f_k(x) \leq f(x) < f_k(x) + \frac{1}{2^{k-1}} \quad \text{for every } x \in I.
\]

Then each \( f_k \) is absolutely measurable. By Lemmas 5 and 7 there is a Baire 1 function \( b_1 \) such that the set of points for which \( b_1(x) = f_1(x) \) contains a \( c \)-dense set \( E \) of type \( F_\sigma \).

Let \( A = \{f_1(x) = f_2(x)\} \cap E \). Then \( A \) is absolutely measurable so that, by Lemma 3, there are pairwise disjoint sets \( A^*, A^{**}, \) and \( A^{***} \) such that \( A = A^* \cup A^{**} \cup A^{***} \) and \( A^* \) is perfectly dense on an open set \( G \) and \( A^* = A \cap G, A^{**} \) is nowhere dense, and \( A^{***} \) is of absolute measure 0.

Let \( H \) be the interior of the complement of \( G \). Then \( H \cap A^* \) is empty. So, \( H \cap A \) is the union of a nowhere dense set and a set of absolute measure zero. If \( H \neq \emptyset \), \( E \cap H \) is a \( c \)-dense \( F_\sigma \) subset of \( H \) since \( E \) is a \( c \)-dense \( F_\sigma \).
subset of $I$. So, by Lemma 8, $(E \cap H) \setminus A$ contains an $F_\sigma$ set which is $c$-dense in $H$. Call this set $F$.

Now, on the complement of $A$ we have

$$f_2(x) = f_1(x) + \frac{1}{2}.$$ 

On $E$, $b_1(x) = f_1(x)$. Since $F \subset E \setminus A$,

$$f_2(x) = b_1(x) + \frac{1}{2} \quad \text{on } F.$$ 

Let $b_2 = b_1 + \frac{1}{2} \chi_H$. Since $H$ is open $\chi_H$ is Baire 1 so that $b_2$ is Baire one. Moreover, $\|b_2 - b_1\| \leq \frac{1}{2}$ where $\|\phi\|$ is the $\sup[|\phi(x)|: x \in I]$ for any function $\phi$ on $I$.

Finally, we note that $b_2(x) = f_2(x)$ on a $c$-dense set of type $F_\sigma$. First, $F$ is an $F_\sigma$ set which is $c$-dense in $H$ and on $F$,

$$b_2(x) = b_1(x) + \frac{1}{2} = f_1(x) + \frac{1}{2} = f_2(x).$$

Next, on $A^*$,

$$b_2(x) = b_1(x) = f_1(x) = f_2(x).$$

But $A^*$ is perfectly dense in $G$ so that it contains an $F_\sigma$ set $K$ which is dense in $G$. Now, $b_2(x) = f_2(x)$ on $F \cup K$ which is a $c$-dense $F_\sigma$ in $G \cup H$, a dense open set in $I$. Thus $b_2(x) = f_2(x)$ on a dense $F_\sigma$ in $I$.

Proceeding by induction, obtain a sequence of Baire 1 functions $\{b_k\}$ such that

$$\|b_{k+1} - b_k\| < \frac{1}{2^k}$$

where $b_{k+1} = f_{k+1}$ on a $c$-dense $F_\sigma$.

We now modify $\{b_k\}$ to obtain $\{g_k\}$, a sequence of Baire one functions which converges uniformly to a Baire 1 function $g$ such that $g(x) = f(x)$ on a $c$-dense $F_\sigma$.

For this purpose, let $\{I_k\}$ be an enumeration of the rational intervals in $[0, 1]$. For each $k$, choose $P_k \subset I_k$ so that

(a) $P_k$ is perfect,

(b) $P_i \cap P_j = \emptyset$ if $i \neq j$,

(c) $b_k(x) = f_k(x)$ on $P_k$.

By Lemma 6, there is a perfect set $Q_k \subset P_k$ such that $f\mid Q_k$ is continuous.

Let

$$g_k(x) = \begin{cases} f(x) & \text{for } x \in Q_1 \cup Q_2 \cup \cdots \cup Q_k, \\ b_k(x) & \text{elsewhere}. \end{cases}$$

Since $Q_1 \cup \cdots \cup Q_k$ is closed, $b_k$ is Baire 1, and $f\mid Q_1 \cup \cdots \cup Q_k$ is continuous, $g_k$ is Baire 1. Now, $|b_{k+1}(x) - b_k(x)| < 1/2^k$ for every $x \in I$,

$g_k(x) = b_k(x)$ on $I \setminus (Q_1 \cup \cdots \cup Q_k)$ and $g_{k+1}(x) = b_{k+1}(x)$ on $I \setminus (Q_1 \cup \cdots \cup Q_k \cup Q_{k+1})$ and $g_k(x) = g_{k+1}(x) = f(x)$ on $Q_1 \cup \cdots \cup Q_k$. Hence

$$\sup[|g_{k+1}(x) - g_k(x)|: x \in I \setminus Q_{k+1}] < 1/2^k.$$
For $x \in Q_{k+1}$, $g_{k+1}(x) = f(x)$ and so

$$|g_k(x) - g_{k+1}(x)| = |g_k(x) - f(x)| = |b_k(x) - f(x)|$$

$$\leq |b_k(x) - b_{k+1}(x)| + |b_{k+1}(x) - f_{k+1}(x)| + |f_{k+1}(x) - f(x)|$$

$$\leq \frac{1}{2^k} + 0 + \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

by (c) and (i). So $\|g_{k+1} - g_k\| < 1/2^{k+1}$.

Since each $g_k$ is Baire 1 and the sequence $\{g_k\}$ converges uniformly, the limit $g$ is also Baire 1 and is equal to $f$ on $\bigcup_k Q_k$, a $c$-dense $F_\sigma$ set. This proves the theorem.

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