NECESSARY AND SUFFICIENT TYPE THEOREM FOR ABSOLUTE NÖRLUND SUMMABILITY OF CONJUGATE SERIES

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Abstract. Sufficient conditions for $|N,p_n|$ summability of the conjugate series of Fourier series have been obtained earlier. The author obtains here a set of necessary and sufficient conditions for the result. This improves earlier results and provides a unified version of them.

1. Introduction and the main results. In the present paper we shall prove a result which is directly connected with an earlier paper due to Dikshit and Kumar [3]. Unless stated otherwise we shall use throughout this paper the same definitions and notations as used in [3]. We shall also use the following additional notations: $B_n(t)$ denotes the conjugate series of $A_n(t)$; $\psi(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \}$; and ‘BV [a, b]’ represents the class of functions of bounded variation over [a, b].

Throughout $\lambda(t)$ will denote a positive nondecreasing function; for $n = 1, 2, 3, \ldots, \lambda(n) = \lambda_n$. Further, we write

$$J_n(\lambda_n) = \sum_{k = n}^{\infty} \left( \frac{P_n \lambda_k}{\lambda_n k P_k} \right).$$

Combining a result due to Bosanquet and Hyslop [1] with a subsequent result due to Mohanty [5], we have the following.

Theorem A. Suppose that

$$t^{\alpha} |\psi(t)| < \infty \text{ for } 0 < \alpha < 1.$$  

Then $\sum_{n=1}^{\infty} n^\alpha B_n(x)$ is summable $|C, \alpha + \delta|$ for every $\delta > 0$ if

$$\int_0^\pi t^{-\alpha - 1} |\psi(t)| \, dt < \infty.$$  

When $\alpha = 0$, then (1.2) is also a necessary condition for the $|C, \delta|$ summability of $\sum_{n=1}^{\infty} B_n(x)$ (see [1, Theorem 4]).

In view of the lemma [5, p. 66] the conditions

$$t^{-\alpha} \psi(t) \in BV[0, \pi]$$
and (1.2), for $0 < \alpha < 1$, are equivalent to the corresponding conditions used by Mohanty in [5, Theorem 2]. However, (1.1') is equivalent to (1.1) whenever (1.2) holds. For, if (1.1) holds, then (cf. [1, p. 495])
\[ \int_0^\pi \left| t^{-\alpha} \psi(t) \right| dt \leq \int_0^\pi \left| t^{-\alpha} d\psi(t) \right| dt + \alpha \int_0^\pi \left| \psi(t) t^{-\alpha-1} \right| dt < \infty \]
and thus (1.1') is valid. Conversely, assuming (1.1'), we have
\[ \int_0^\pi \left| t^{-\alpha} d\psi(t) \right| dt \leq \int_0^\pi \left| t^{-\alpha} \psi(t) \right| dt + \alpha \int_0^\pi \left| \psi(t) t^{-\alpha-1} \right| dt < \infty \]
and, therefore, (1.1) is valid.

The sufficiency part of Theorem A was generalised in the form of the following results (see [6], [2] respectively).

**Theorem B.** Suppose (1.1) and (1.2) hold with $\alpha = 0$, and let \( \{ p_n \} \) be a nonnegative nonincreasing sequence such that
\[ \sum_{n=1}^\infty |\Delta R_n| < \infty \quad \text{and} \quad \sum_{n=1}^\infty |\Delta S_n| < \infty, \]
where \( R_n = (n+1)p_n/p_n \) and \( S_n = \sum_{k=1}^n p_k/kP_n \), then \( \sum_{n=1}^\infty B_n(x) \) is summable \( \|N, p_n\| \).

**Theorem C.** Let \( \{ p_n \} \) be such that \( p_n > 0, p_{n+1}/p_n < p_{n+2}/p_{n+1} \leq 1 \) for all \( n \). Then, for $0 < \alpha < 1$, \( \sum_{n=1}^\infty n^\alpha B_n(x) \) is summable \( \|N, p_n\| \), if (1.1), (1.2) hold, and \( J_n(n^\alpha) < K \).

Considering a class of functions \( \lambda(t) \), we shall show that a condition which in particular reduces to (1.2) of Theorem A, is a necessary condition for \( \|N, p_n\| \) summability of \( \sum_{n=1}^\infty \lambda_n B_n(x) \). As for the sufficiency part of the theorem, we obtain a sharper result which includes both Theorem B and Theorem C. We shall prove the following.

**Theorem.** Suppose that
\[ (1.3) \quad \int_0^\pi \lambda(C/t)|d\psi(t)| dt < \infty, \quad \text{where } C \text{ is a number } > \pi; \]
and let \( \{ p_n \} \) be a nonnegative nonincreasing sequence such that
\[ (1.4) \quad J_n(\lambda_n) < K, \quad n = 1, 2, \ldots, \]
then \( \sum_{n=1}^\infty \lambda_n B_n(x) \) is summable \( \|N, p_n\| \), if and only if,
\[ (1.5) \quad \int_0^\pi \lambda(C/t)|\psi(t)|t^{-1} dt < \infty. \]

If we assume \( \psi(0) = 0 \), then \( \psi(t) = \int_0^t d\psi(u), \) so that by a change in order of integration we get for $\alpha > 0$,
\[ \int_0^\pi t^{-\alpha-1}|\psi(t)| dt \leq \int_0^\pi t^{-\alpha-1} dt \int_0^t |d\psi(u)| < \frac{1}{\alpha} \int_0^\pi u^{-\alpha}|d\psi(u)|. \]
Thus, if \( \psi(0) = 0 \) and $\alpha > 0$, then (1.1) implies (1.2). However, it may be observed that (1.3) does not imply (1.5) even if \( \psi(0) = 0 \). For if we take
\( \lambda(u) = \log u \) and \( \psi(t) = (\log(C/t))^{-2} \), then (1.3) is true but (1.5) is not satisfied.

2. Preliminary results. We need the following lemmas for proof of the theorem. We introduce the following notations for convenience: \( \tau = [\pi/r] \); for each positive integer \( n \), \( P(n, k) = P_{n-k}/P_n - P_{n-k-1}/P_{n-1} \), where \( 0 \leq k \leq n \), \( P(0, 0) = 1 \).

**Lemma 1.** Let \( \{p_n\} \) be a nonnegative nonincreasing sequence. Then (i) for all \( k \geq 0 \), \( \sum_{n=a}^{b} P(n, k) \leq 1 \), where \( 1 \leq a \leq b < \infty \), and \( P(n, k) \geq 0 \);

(ii) \( \sum_{k=0}^{n} P(n, k) \geq 1/2 \) for all \( n > 0 \).

**Proof.** The result (i) of the lemma is contained in ([3, Lemma 3]). Writing \( P_n = \sum_{k=0}^{n} p_k \), we have

\[
\sum_{k=0}^{n} P(n, k) = \frac{P_n}{P_n} - \frac{P_{n-1}}{P_{n-1}} = 1 - \frac{P_{n-1}P_n}{P_{n-1}P_n}.
\]

In view of this equality, result (ii) is equivalent to

\[
(2.1) \quad p_n P_{n-1} < \frac{1}{2} P_{n-1} P_n,
\]

which we now prove by induction. The result (2.1) is trivially true for \( n = 1 \). Assuming that the result (2.1) is true for \( n \), we obtain the following by adding \( p_n P_n \) to both the sides of (2.1).

\[
(2.2) \quad p_n P_n < \frac{1}{2} P_n (P_n + p_n).
\]

Since \( p_{n+1} - p_n < 0 \) and \( P_n > \frac{1}{2} P_n \), we have

\[
(2.3) \quad (p_{n+1} - p_n)P_n < \frac{1}{2} P_n (p_{n+1} - p_n).
\]

Adding (2.2) and (2.3), we see that (2.1) holds with \( n \) replaced by \( (n + 1) \). Hence the result.

**Lemma 2.** If \( \{p_n\} \) be nonnegative nonincreasing and (1.4) holds, then (i) \( \lambda_{2n} < K\lambda_n \), (ii) for \( 0 < k < n \), \( k\lambda_n \leq K\lambda_k \) and (iii) \( \sum_{k=1}^{\infty} (P_n - P_{n-k})/k \leq K P_n \).

**Proof.** It is clear that (ii) includes (i) which is contained in [3, Lemma 4]. We, therefore, consider (ii). Since the nonnegative nonincreasing nature of \( \{p_n\} \) implies that \( \{P_n/n\} \) is nonincreasing, we have from (1.4)

\[
\frac{K\lambda_k}{k} > \sum_{r=k}^{\infty} \frac{\lambda_r}{r^2} > \sum_{r=n}^{\infty} \frac{\lambda_r}{r^2} > \lambda_n \sum_{r=n}^{\infty} \frac{1}{r^2},
\]

since \( \{\lambda_n\} \) is nondecreasing. Thus, (ii) follows.

In order to prove (iii) we first observe that \( S_n < K \) is equivalent to \( J_n(1) < K \) (see [4, Lemma 4]) which is a direct consequence of (1.4). The result now follows when we observe that for nonnegative nonincreasing sequence \( \{p_n\} \), \( P_n - P_{n-k} < P_k \).
Lemma 3. If \( \{ p_n \} \) be nonnegative nonincreasing, and \( (1.4) \) holds, then uniformly for \( 0 < t < \pi \)
\[
\sum_{n=2}^{\infty} \left| \sum_{k=\tau+1}^{n} p(n, k) k^{-1} \lambda_k \exp(ikt) \right| \leq K\lambda(C/t),
\]
where \( \tau = [\pi/t] \).

The result of Lemma 3 follows from the proof of the result \( \Sigma^* \leq K\lambda(C/t) \), as given in ([3, Proof of Theorem 1, (4.6)]).

Lemma 4. Suppose that \( (1.3) \) holds, and let for \( n > 1 \), \( E_n = \{ t : |\psi(t)| \) is continuous at \( t \in [\pi/n, \pi/(n-1)] \} \). Then there exist points \( \theta_n \) and \( \theta'_n \) in \( E_n \) such that, for all \( x \in E_n \),
\[
(2.4) \quad |\psi(\theta_n)| - 1/n < |\psi(x)| < |\psi(\theta'_n)| + 1/n.
\]
Further, (i) if \( (1.4) \) holds and \( \{ p_n \} \) is nonnegative nonincreasing, then \( (1.5) \) implies \( \Sigma_{n=2}^{\infty} \lambda_n |\psi(\theta_n)/n < \infty \), and \( \Sigma_{n=2}^{\infty} \lambda_n |\psi(\theta'_n)/n < \infty \) implies \( (1.5) \); and (ii) \( \Sigma_{n=2}^{\infty} \lambda_n |\Delta \psi(\theta_n)| < \infty \).

Proof. It is clear that \( (1.3) \) implies that \( \psi(t) \in BV[0, \pi] \) so that \( |\psi(t)| \in BV[0, \pi] \) and thus \( (2.4) \) follows directly. Further, since \( |\psi(t)| \) possesses at most countable number of ordinary discontinuities, we have by the result (i) of Lemma 2
\[
\frac{|\psi(\theta_n)| \lambda_n}{n} \leq K \int_{E_n} \left( |\psi(t)| + 1/n \right) \frac{\lambda(\pi/t)}{t} \, dt,
\]
which gives that
\[
\sum_{n=2}^{\infty} \frac{|\psi(\theta_n)| \lambda_n}{n} \leq K \int_{0}^{\pi} \frac{|\psi(t)| \lambda(\pi/t)}{t} \, dt + K \sum_{n=2}^{\infty} \frac{\lambda_n}{n^2}.
\]
Applying the condition \( (1.4) \) along with the fact that \( \{ P_n/n \} \) is nonincreasing, we obtain the first result in (i). The second one can be obtained similarly.

In order to prove (ii), we apply result (i) of Lemma 2 so that
\[
\lambda_n |\Delta \psi(\theta_n)| \leq K\lambda_{n-1} \left| \int_{\theta_n}^{\theta_{n+1}} d\psi(t) \right| \leq K \int_{\theta_n}^{\theta_{n+1}} \lambda(C/t) |d\psi(t)|,
\]
since \( \lambda(t) \) is nondecreasing. This proves (ii).

3. Proof of the Theorem. We have
\[
B_k(x) = \frac{2}{\pi} \int_{0}^{\pi} \psi(t) \sin kt \, dt.
\]
We first integrate by parts and then break the range of integration in \( (0, \theta_n) \) and \( (\theta_n, \pi) \) where \( \theta \)'s are the points introduced in Lemma 4. Thus, we have
\[ \frac{k \pi}{2} B_k(x) = - \int_0^\pi \left( 1 - \cos kt \right) d\psi(t) \]

\[ = - \int_0^{\theta_n} \left( 1 - \cos kt \right) d\psi(t) - \int_0^\pi d\psi(t) + \int_0^\pi \cos kt \ d\psi(t) \]

\[ = - 2 \int_0^{\theta_n} \sin^2 \left( \frac{kt}{2} \right) d\psi(t) + \psi(\theta_n) + \int_0^\pi \cos kt \ d\psi(t). \]

Writing \( V_n(a) = t_n - t_{n-1} \), where \( t_n \)'s are \( (N, p_n) \) means of \( a = \sum_{n=1}^\infty \lambda_n B_n(x) \), we have

\[ -\Sigma_1 + \Sigma_2 - \Sigma_3 < \frac{\pi}{2} \sum_{n=2}^\infty |V_n(a)| < \Sigma_1 + \Sigma_2 + \Sigma_3, \]

where

\[ \Sigma_1 = 2 \sum_{n=2}^\infty \left| \int_0^{\theta_n} \left\{ \sum_{k=1}^n \frac{P(n, k)\lambda_k \sin^2(kt/2)}{k} \right\} d\psi(t) \right|, \]

\[ \Sigma_2 = \sum_{n=2}^\infty \left| \psi(\theta_n) \right| \sum_{k=1}^n \frac{P(n, k)\lambda_k}{k}, \]

\[ \Sigma_3 = \sum_{n=2}^\infty \left| \int_{\theta_n}^\pi \left\{ \sum_{k=1}^n \frac{P(n, k)\lambda_k \cos kt}{k} \right\} d\psi(t) \right|. \]

We first show that \( \Sigma_1 \) and \( \Sigma_3 \) are bounded. Since for the integral in \( \Sigma_1 \), \( t < \theta_n < \pi/(n-1) \), we have

\[ \Sigma_1 < 2 \int_0^\pi \left| d\psi(t) \right| \left( \sum_{n=2}^{\tau+1} \sum_{k=1}^n \frac{P(n, k)\sin^2(kt/2)\lambda_k}{k} \right). \]

Observing that \( |\sin nt| < nt \), and applying results (i) of Lemma 1, Lemma 2, we have

\[ \sum_{n=2}^{\tau+1} \sum_{k=1}^n \frac{P(n, k)\sin^2(kt/2)\lambda_k}{k} \]

\[ \leq t^2 \sum_{k=1}^{\tau+1} k\lambda_k \sum_{n=k}^{\tau+1} P(n, k) < K\lambda_{\tau+1} < K\lambda(C/t). \]

Hence, by virtue of the hypothesis (1.3),

\[ (3.2) \Sigma_1 < \infty. \]

Now, we consider \( \Sigma_3 \). We write

\[ g_n(t) = \sum_{k=1}^n P(n, k)\lambda_k k^{-1} \cos kt \quad \text{for} \ \theta_n < t < \pi, \]

otherwise zero. We have

\[ \Sigma_3 < \sum_{n=2}^\infty \int_0^\theta \left| g_n(t) \right| \left| d\psi(t) \right|. \]
In order to prove that $\Sigma_3 < \infty$, we first demonstrate that
\[
\Sigma' = \sum_{n=\tau}^{\infty} \left| \sum_{k=1}^{n} P(n, k) \lambda_k k^{-1} \exp(ikt) \right| \leq K\lambda(C/t),
\]
uniformly in $0 < t < \pi$.

Breaking the range of summations suitably and observing that $|\exp(ikt)| = 1$, we have
\[
\Sigma' \leq \sum_{n=\tau}^{\infty} \sum_{k=1}^{\tau} P(n, k) \lambda_k / k + \sum_{n=\tau+1}^{2\tau+1} \sum_{k=\tau+1}^{n} P(n, k) \lambda_k / k
\]
\[+ \sum_{n=2\tau+2}^{\infty} \left| \sum_{k=\tau+1}^{n} P(n, k) \lambda_k k^{-1} \exp(ikt) \right|.
\]

Using result (i) of Lemma 1, Lemma 2, together with the fact that $\{\lambda_n\}$ is nondecreasing, and Lemma 3, we have
\[
\Sigma' \leq \sum_{k=1}^{\tau} \lambda_k / k \left( 1 - \frac{P_{\tau-k-1}}{P_{\tau-1}} \right) + \sum_{k=\tau+1}^{2\tau+1} \frac{\lambda_k}{k} \sum_{n=k}^{2\tau+1} P(n, k) + K\lambda(C/t)
\]
\[\leq K\lambda(C/t) + K\lambda(C/t) \leq K\lambda(C/t).
\]

This result, by virtue of the hypothesis (1.3) and the fact that $\gamma_n(t) = 0$ when $n < \tau$, yields that
\[
(3.3) \quad \Sigma_3 < \infty.
\]

Evidently,
\[
(3.4) \quad |V_1(a)| = P(1, 1)|B_1(x)| < \infty.
\]

In view of (3.1), (3.2), (3.3) and (3.4) it is enough for proving the theorem to show that $\Sigma_2 < \infty$, if and only if (1.5) holds. In order to establish the sufficiency part of the theorem, we first observe that $\psi(t) \to 0$ as $t \to 0$. For if $\psi(t) \to l \neq 0$, then in view of the fact that the positive nondecreasing property of $\lambda(t)$ implies the existence of a positive constant $a$ such that $\lambda(C/t) > a$ in some right hand neighbourhood of the origin, it follows that (1.5) is contradicted. Now effecting suitable changes in order of summations and applying Abel’s transformation, we get
\[
\Sigma_2 = \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} |\psi(\theta_n)| P(n, k) + O(1)
\]
\[= \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \Delta|\psi(\theta_n)| \frac{P_{n-k}}{P_n} + O(1)
\]
\[= \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \Delta|\psi(\theta_n)| - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \frac{P_n - P_{n-k}}{P_n} \Delta|\psi(\theta_n)| + O(1)
\]
\[= \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |\psi(\theta_k)| - \sum_{n=2}^{\infty} \Delta|\psi(\theta_n)| \sum_{k=2}^{n} \frac{(P_n - P_{n-k})\lambda_k}{kP_n} + O(1) < \infty,
\]
by result (iii) of Lemma 2, Lemma 4 and the fact that \( (\lambda_n) \) is nondecreasing. This completes the proof of the sufficiency part of the theorem.

Next, by the results (ii) of Lemmas 1 and 2, we have

\[
\sum_2 > K \sum_{n=2}^{\infty} \frac{\psi(\theta_n) |\lambda_n|}{n}
\]

\[
> -K \sum_{n=2}^{\infty} \|\psi(\theta_n)\| - \|\psi(\theta'_n)\| \frac{\lambda_n}{n} + K \sum_{n=1}^{\infty} \frac{\psi(\theta'_n) |\lambda_n|}{n}.
\]

Since by result (ii) of Lemma 2, \( \lambda_n/n \leq K \), the first term on the right hand side is bounded by virtue of the fact that \( |\psi(t)| \in BV[0, \pi] \), which is a consequence of (1.3). Now the necessity part of the theorem follows when we use result (i) of Lemma 4.

This completes the proof of the theorem.

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References


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