LEFT AND RIGHT INVARIANCE IN AN INTEGRAL DOMAIN

RAYMOND A. BEAUREGARD

Abstract. A ring is said to be right (left) invariant if each of its right (left) ideals is twosided. In this paper we resolve the conjecture: Every right invariant integral domain which satisfies the left Ore (multiple) condition is left invariant. A proof is given for the class of LCM domains satisfying a finiteness condition. An example is given to show that the LCM hypothesis cannot be dropped. A second example shows that the conjecture fails even in a Bezout domain which does not have the finiteness condition. The problem of right versus left boundedness is also considered.

An integral domain is said to be right (left) invariant if each of its right (left) ideals is twosided. This paper is motivated by the following question: [3, p. 162]: Is every right invariant integral domain which is assumed to be left Ore (intersection of any two nonzero left ideals is nonzero) also left invariant? We show that the answer is affirmative for LCM domains satisfying a finiteness condition, but is otherwise false. Other related questions (dealing with boundedness) are discussed.

In what follows R is an integral domain, i.e. a ring with unity which is free of proper divisors of zero. The definitions referred to above may be phrased completely in terms of principal ideals. In particular, if \( Ra \subseteq aR \) then the element \( a \) of \( R \) is said to be right invariant; \( R \) is right invariant if each of its elements is. A similar statement holds for left invariance. If an element or a ring is both right and left invariant it is said to be invariant.

A right (left) LCM domain is a ring in which the intersection of any two principal right (left) ideals is again principal; an LCM domain is a ring that has both properties. If \( 0 \neq ab' = ba' \in R \) in a right LCM domain \( R \), then \( aR \cap bR \) is generated by an element which is a least common right multiple of \( a' \) and \( b' \), denoted \( [a, b]_r \); in addition, the greatest common right factor \( (a', b')_r \) of \( a' \) and \( b' \) exists. Similar remarks apply for left LCM domains. The following lemma, which is proved in [1], gives the relationship of these terms.

**Lemma 1.** Let \( R \) be an LCM domain and let \( 0 \neq ab' = ba' \in R \). Then

\[
ab' = ba' = [a, b]_r(a', b')_r = (a, b)[a', b']_r.
\]

We shall also need the following easy result.

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Lemma 2. If $0 \neq [a, b]_a = ba'$ in a ring $R$ and if $a$ is right invariant, then $aR \subseteq a'R$.

**Proof.** By right invariance we have $ba \in aR \cap bR = ba'R$.

Theorem 1. Let $R$ be an LCM domain which satisfies the ascending chain condition (acc) for either principal right or principal left ideals or which is atomic. If $R$ is left Ore and right invariant then $R$ is left invariant.

**Proof.** First we show that each atom (i.e. irreducible element) $a \in R$ is invariant. For any $r \in R$ let $xar = ya$ be a generator of $Rar \cap Ra$. The left Ore condition assures that this term is not zero so Lemma 1 applies and $(x, y)_a = 1$. If $d = (ar, a)$, then either $Rd = R$ or $Rd = Ra$ because $a$ is an atom. In the first case we have $[x, y]_a = xar = ya$ by Lemma 1 and so $xR \subseteq aR$, $yR \subseteq arR \subseteq aR$ by Lemma 2 in contradiction with $(x, y)_a = 1$. Therefore $Rd = Ra$ and $ar \in Ra$. This shows that $a$ is left invariant and hence invariant.

Next we show that $R$ is atomic, i.e. each nonzero nonunit is a product of atoms. Suppose not. Using the left acc we may choose $Rx$ maximal in $\{Rx|x \text{ is not a product of atoms}\}$, and choose an atom $a$ such that $Rx \subseteq Ra$. Then $x \in aR$ (by invariance of $a$), i.e. $x = ax_1$ for some $x_1$ which cannot be a product of atoms; thus $Rx \subseteq Rx_1$ is a contradiction. A similar argument shows that $R$ is atomic if $R$ satisfies the right acc. Each nonzero nonunit is a product of (invariant) atoms and is consequently invariant.

We single out the following special case of Theorem 1. Recall that a principal right ideal domain (PRI domain) has the acc for right ideals and is a right Ore domain; in addition, it is a weak Bezout domain (= 2-fir), and hence an LCM domain (cf. [3, pp. 47–50]). Theorem 1 and its left-right analog imply the following:

**Corollary 1.** In a PRI domain the following two conditions are equivalent:

(i) $R$ is left Ore and right invariant.

(ii) $R$ is left invariant.

We give two examples to show the necessity of the hypotheses in Theorem 1.

**Example 1 (A non-LCM domain with both chain conditions).** Let $F$ be a (commutative) field extension of a field $K$ with an automorphism $\sigma: F \rightarrow F$ which maps $K$ into a proper subfield of $K$. (For example, $F = Q(t_1, t_2, t_3, \ldots)$, $K = Q(t_1, t_3, t_5, \ldots)$, and $\sigma$ the $Q$-automorphism of $F$ defined by $\sigma(t_{2n-1}) = t_{2n+1}$, $\sigma(t_{2n+2}) = t_{2n}$ for $n = 1, 2, \ldots$ and $\sigma(t_2) = t_1$.)

Let $P$ be the skew formal power series ring

$$P = F[[x, \sigma]] = \left\{ \sum_{i=0}^{\infty} x^i a_i | a_i \in F \right\}$$

in which multiplication is defined by $ax = xa^\sigma$ ($a \in F$). It is easy to show that $P$ is a local PRI and PLI domain (cf. [5]). Let $R = \{ f(x) \in P | f(0) \in F \}$. Let $R = \{ f(x) \in P | f(0) \in F \}$.
$K$) be the subring of $P$ consisting of all power series with constant term in $K$. Clearly $R$ is atomic; in fact, it is easily verified that $R$ has the acc for both principal right and principal left ideals. In addition, $R$ is left and right Ore because $P$ has this property.

To show that $R$ is right invariant we first observe that if $h$ is any unit in $P$ and $g \in R$ then $h^{-1}gh \in R$. Now let $f \in R$ be a nonunit written in the form $f = x^n h$ where $h(0) \neq 0$ (so that $h$ is a unit in $P$) and $n > 0$. Clearly $x$ is right invariant in $R$. Also, $xh$ is right invariant, for if $r \in R$ and $r'$ is chosen in $R$ so that $rx = x r'$, then $rxh = xh(h^{-1}r'h) \in xhR$. Thus $f$ is right invariant. Finally we note that $x$ is not left invariant because $\sigma[K] \neq K$; therefore the only nonzero members of $R$ that are left invariant are the units.

**Example 2 (A nonatomic Bezout domain).** Let $K$ be a commutative Bezout domain with quotient field $F$ and with monomorphism $\sigma : K \to K$ which is not an epimorphism but which when extended to $F$ in the usual way is an isomorphism. (For example we can take $K$ to be the principal ideal domain of formal Laurent series $Z\langle\langle t \rangle\rangle$ over the ring of integers $Z$ and take $\sigma$ to be the $Z$-monomorphism defined by $\sigma(t) = 2t$.)

Let $P$ be the ring of skew formal power series, as above, in which multiplication is defined by $ax = xa$ ($a \in F$). Let $R = \{f(x) \in P | f(0) \in K\}$. That $R$ is a right and left Bezout domain follows from the proposition below.

We check the following.

(i) Each $a \in K$ is right and left invariant.

First observe that if $0 \neq a \in K$ and $f \in R$ then $a f a^{-1} \in R$. Thus $fa = a^{-1}f$ and, similarly, $af \in Ra$.

(ii) Each $f \in R$ is right invariant.

For $f$ may be written $f = x^n u a$ where $u$ is a unit power series in $R$ with $u(0) = 1$ and $a, s \in K$; then $fs$ is right invariant in $R$ (being a product of the right invariant elements $x^n, u, a$); since $s$ is invariant it follows that $f$ is right invariant.

(iii) The left invariant elements of $R$ have the form $f = ub$, where $b \in K$ and $u$ is a unit in $R$.

First observe that $x$ is not left invariant because $\sigma[K] \neq K$. If $f = x^n u a$ written as in (ii) is left invariant then so is $fs = x^n ua$ (because $s$ is invariant); it then follows that $x^n$ is left invariant because $ua$ is invariant; contradiction with $\sigma[K] \neq K$ is avoided only if $n = 0$ and $f = u a$ since $u(0) = 1$ we have $as^{-1} \in K$ as desired.

**Proposition.** Let $P = F[[x, a]]$ and $R = \{f(x) \in P | f(0) \in K\}$ be the rings constructed above. Then $R$ is a right and left Bezout domain.

**Proof.** To show that $R$ is right Bezout let $f, g \in R$. First assume that $f = a + f_1, g = b + g_1$ where $a, b \in K$ and $a \neq 0$. Let $dK = aK + bK$. Then since $f_1 = d^{-1}f_1$, $g = d^{-1}g_1$, we have $fR, gR \subset dR$. To show the reverse inclusion we observe that $fh = 1$ for some $h \in P$ and
so $c = f(hc) \in fR$ for some $0 \neq c \in K$. Thus $a = f - f_1 = f - c(c^{-1}f_1) \in fR + gR$ and $b = g - g_1 = g - c(c^{-1}g_1) \in fR + gR$ which shows $dR \subseteq fR + gR$.

Now let $f = x^n(a_1b_1^{-1} + h_1) = x^{n_1}f_1$, $g = x^{n_2}(a_2b_2^{-1} + h_2) = x^{n_2}g_1$ where $h_1 \in R$ and $a_i, b_i \neq 0$ in $K$. We assume that $0 < n_1 < n_2$. If $n_1 = n_2$ then $bf_1R + bg_1R = dR$, where $b = b_1b_2$ and $dK = a_1b_2K + a_2b_1K$. Multiplying on the left by $x^{-n_2}$ we find $fR + gR = x^{-n_2}dR$, a principal right ideal of $R$. If $n_1 < n_2$ then $g = x^{n_1}(f_1^{-1}x^{n_2-n_1}g_2) \in fR$ so that $fR + gR = fR$.

We have shown that $R$ is a right Bezout domain. Since $P$ is a left and right Ore domain the same is true of $R$. Consequently, $R$ is a left and right Bezout domain.

A ring is said to be right (left) bounded if each nonzero right (left) ideal contains a nonzero twosided ideal. Again the definition may be phrased in terms of principal one and twosided ideals. The example just given is a Bezout domain (hence left Ore) which is right invariant (hence right bounded) but not left bounded. The more difficult question of whether a ring satisfying one of the finiteness conditions of Theorem 1 which is left Ore and right bounded is also left bounded is open even for the special case of a PRI domain (however see Corollary 2 below). If $R$ is a right bounded PLI domain then $R$ is left bounded. For in this case a right invariant element $a \in R$ gives rise to a twosided ideal of the form $aR$ which must be principal as a left ideal by hypothesis. From this it follows easily that $aR = Ra$ (cf. [4, p. 37]). In fact, in this case $R$ is a PRI domain. We summarize in the following:

**Theorem 2.** A right bounded PLI domain is both left bounded and a PRI domain.

**Proof.** It remains to prove the second assertion. Since a right Ore PLI domain is a right Bezout domain it suffices to show that $R$ is atomic. Let $a \in R$ be a nonzero nonunit and let $a^*$ be its right bound (see [2] for the definition). We have noted above that all right invariant elements such as $a^*$ must be invariant. Using the left acc we may write $a^* = a_1 \cdots a_n$, where each $a_i$ is invariant but has no proper invariant factors. If $p_i$ is an atomic factor of $a_i$, then $a_iR = p_i^*R$ where $p_i^*$ is the right bound of $p_i$. By [2, Theorem 3.2] each $p_i^*$ is a product of atoms (similar to $p_i$). Thus $a^*$ and, consequently, $a$ is a product of atoms.

**Corollary 2.** Let $R$ be an atomic weak Bezout domain. If $R$ is left Ore and right bounded then $R$ is left bounded.

The converse in Corollary 2 (or in Theorem 1) does not hold as shown by the ring of skew formal power series $F[[x, \sigma]] = \{\Sigma_{n=0}^{\infty} q_n x^n | q_n \in F\}$ in which multiplication is defined by $xa = a^n x$ for a monomorphism $\sigma$ on the field $F$. This ring is a left invariant PLI domain which is not right bounded (indeed not right Ore) if $\sigma$ is not an epimorphism.
We change sides and restate Theorem 2 for the sake of comparing with Corollary 1.

**Corollary 3.** A left bounded PRI domain is a right bounded PLI domain.

Whether “left bounded” may be replaced by the weaker “left Ore right bounded” in Corollary 3 is open. Indeed, whether the proposed hypothesis is actually weaker in a PRI domain is open. In an atomic PRI domain the two hypotheses are equivalent.

**References**