A BOUNDARY VALUE PROBLEM FOR $H^\infty(D)$

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Abstract. Let $W = \bigcup_{n=1}^{\infty} W_n$ be an $F_\sigma$ subset of the unit circle of measure 0 and let $(q_n), n > 1$, be a decreasing sequence with $q_1 < 1$ and $\lim_n q_n = 0$. There exists an $H$ in $H^\infty(D)$ of norm $q_1$ whose modulus has radial limit along every radius which has radial limit of modulus $q_1$ on $W_1$ and $q_{n+1}$ on $W_{n+1} \setminus \bigcup_{k=1}^{n} W_k$. If $W$ is simultaneously a $G_\sigma$ set, $H$ may be chosen to have no zeros on $C$. It follows that for $W$ countable, say $W = \{e^{i\theta_n}\}, n > 1$, there is such an $H$ of norm 1 for which $\lim_{r \to 1} H(re^{i\theta_n}) = 1/n$. The proof of the theorem depends on the existence of a special collection of closed sets $(S_\lambda), \lambda > 1$, real, for which the function $h$, defined by $h(x) = a_n + (\inf\{A|x \in S\}) - n(a_{n+1} - a_n), a_n = -\ln q_n$, is such that the function $H(w) = \exp(-1/2\pi) \int [(w + e^{i\theta})/(e^{i\theta} - w)]h(u) \, du$ has the required properties. Some of the techniques used are similar to those developed in an earlier paper [1].

Notation. Let $I$ be a fixed interval on the real line. For each Lebesgue measurable subset $E$ of $I$, $|E|$ denotes its Lebesgue measure. $E'$ denotes the complement of $E$ in $I$.

The letters $k, m, n$ and $s$ when they occur as symbols denote natural numbers. $C$ denotes the unit circle and $D$ the unit disc.

Purpose. The aim of this work is to prove the following theorem.

Theorem. Let $W = \bigcup_{n=1}^{\infty} W_n$ be an $F_\sigma$ subset of the unit circle of measure 0 and let $(q_n), n > 1$ be a strictly decreasing sequence with $q_1 < 1$ and $\lim_n q_n = 0$. There exists a function $H$ in $H^\infty(D)$ of norm $q_1$ whose modulus has radial limit along every radius which has radial limit of modulus $q_1$ on $W_1$ and $q_{n+1}$ on $W_{n+1} \setminus \bigcup_{k=1}^{n} W_k$. If $W$ is simultaneously a $G_\sigma$ set, then $H$ may be chosen to have no zeros on $C$.

An immediate corollary of the theorem is that for $W$ countable, say $W = \{e^{i\theta_n}\}, n > 1$, there is such an $H$ of norm 1 for which $\lim_{r \to 1} H(re^{i\theta_n}) = 1/n$.

Preliminary lemmas. Three lemmas precede the proof of the theorem. In addition, frequent reference is made to a lemma of Lusin-Menchoff. For convenience this lemma is stated first.

LUSIN-MENCHOFF LEMMA. Let $M_1$ be an arbitrary nonempty measurable subset of $(0, 1).$ Let $M_2$ be a closed subset of $M_1$ consisting only of points of
density of $M_1$. Then for every positive number $p$ there exists a closed set $M_p$ with $M_2 \subset M_p \subset M_1$ satisfying:

1. Every point of $M_2$ is a point of density of $M_p$ and every point of $M_p$ is a point of density of $M_1$.
2. $|M_p| > |M_2| + (1 - 2^{-2-p})(|M_1| - |M_2|)$.
3. If $x \in M_2$ and $|M_1 \cap (x, x + t)|/|t| > 1 - \varepsilon$ for $|t| < 1/m$, then $|M_p \cap (x, x + t)|/|t| > 1 - \varepsilon - 2^{-m-p+1}$, in particular, if $|M_1 \cap (0, 1)| = 0$, $|M_p \cap (x, x + t)|/|t| > 1 - 2^{-m-p+1}$ for $|t| < 1/m$.

A proof of this lemma may be found in [2]. It is clear that the interval $(0, 1)$ can be replaced by an arbitrary open interval $(a, b)$. The interval may also be closed or half-open provided suitable one-sided restrictions are imposed in (1) and (3) at included endpoints. In the lemmas that follow such restrictions are assumed without explicit mention.

**Lemma 1.** Let $W = \bigcup_{n=1}^{\infty} W_n$, $W_n$ closed, be an $F_\alpha$ set of measure 0 contained in $[a, b]$. There exists an increasing sequence of closed sets $\{S_n\}_{n \geq 1}$ for which

1. $|S_n| > L(1 - 1/2^n)$ for all $n$, where $L = b - a$.
2. $W_1 \subset S_1$ and $W_n \cup \bigcup_{k=1}^{n-1} W_k \subset S_n \setminus S_{n-1}$ for $n > 1$.
3. For each $w \in W_n \setminus \bigcup_{k=1}^{n-1} W_k$, $n > 1$, there exists a $T_w > 0$ such that $|S_n \cap (w, w + t)| = 0$ for all $t$ with $|t| < T_w$.
4. For each $n$ there exists a constant $P_n > n$ such that $|S_{n+1} \cap (x, x + t)|/|t| > 1 - 2^{-m-P_n+1}$ whenever $x \in S_n$ and $|t| < 1/m$.

If $W$ is simultaneously a $G_\delta$ set, then the $S_n$ may be chosen so that, in addition, $\bigcup_n S_n = [a, b]$.

**Proof.** Set $L = b - a$. If $W$ is also a $G_\delta$ set, then $W'$ is an $F_\alpha$ set and may be written as $W' = \bigcup_{n=1}^{\infty} F_n$, $F_n$ closed.

Let $P_1$ be an arbitrary closed subset of $W'$ with $|P_1| > L(1 - \frac{1}{2})$ and set $S_1 = P_1 \cup W_1$. If $W$ is a $G_\delta$ set, set $S_1 = P_1 \cup W_1 \cup F_1$. Let $p_1$ be a number greater than 1 for which $|S_1| + (1 - 2^{-2-p_1})(L - |S_1|) > L(1 - 1/2^2)$.

Since $S_1$ is a subset of $W' \cup W_1$, there exists a closed set $P_2$ with $S_1 \subset P_2 \subset W' \cup W_1$ satisfying properties (1)–(3) of the Lusin-Menchoff Lemma with $M_1 = W' \cup W_1$, $M_2 = S_1$ and $p = p_1$.

Set $S_2 = P_2 \cup W_2 \setminus W_1$. Since $W_1$ is a subset of $S_1$ and hence, of $P_2$, $S_2 = P_2 \cup W_2$, which is closed. If $W$ is also a $G_\delta$ set, set $S_2 = P_2 \cup F_2 \cup W_2 \setminus W_1$.

Continue defining sets $S_n$, $n > 2$, inductively as follows. Let $p_{n-1}$ be a number greater than $n - 1$ for which $|S_{n-1}| + (1 - 2^{-2-p_{n-1}})(L - |S_{n-1}|) > L(1 - 1/2^n)$.

By the Lusin-Menchoff Lemma, there is a closed set $P_n$ with $S_{n-1} \subset P_n \subset W' \cup \bigcup_{k=1}^{n-1} W_k$ satisfying (1)–(3) with $M_1 = W' \cup \bigcup_{k=1}^{n-1} W_k$, $M_2 = S_{n-1}$ and $p = p_{n-1}$. Set $S_n = P_n \cup W_n \cup \bigcup_{k=1}^{n-1} W_k$. If $W$ is a $G_\delta$ set, set $S_n = P_n \cup F_n \cup W_n \cup \bigcup_{k=1}^{n-1} W_k$. The sequence $\{S_n\}$, $n \geq 1$, has all the required properties.
Property (b) follows immediately from the construction of the \( S_n \). Property (a) holds since \(|S_n| > |S_{n-1}| + (1 - 2^{-2^{-n-1}})(L - |S_{n-1}|), n > 1\), by (2) of the Lusin-Menchoff Lemma, and the right side of the inequality is greater than \( L(1 - 1/2^n)\) by choice of \( p_n \). If \( w \in W_n \setminus \bigcup_{k=1}^{n-1} W_k \), \( n > 1\), then \( w \) is not in the closed set \( P_n \) (or in \( P_n \cup F_n \)) since \( P_n \subset W' \cup \bigcup_{k=1}^{n-1} W_k \). Let \( T_w \) be the distance from \( w \) to \( P_n \) (or to \( P_n \cup F_n \)). Then for \( |t| < T_w \), \( |S_n \cap (w, w + t)| = |W_n \setminus \bigcup_{k=1}^{n-1} W_k \cap (w, w + t)| = 0 \). Thus (c) holds. Finally, if \( W' = \bigcup_{n=1}^{\infty} F_n \), \( F_n \) closed, \( S_n \) is chosen so that \( F_n \subset S_n \). Consequently, \( \bigcup_n S_n \) contains both \( W \) and \( W' \) and must equal \([a, b]\). Q.E.D.

It should be noted that an additional restriction may be imposed on the \( p_k \), which will be used later. If \( \{a_k\}, k > 1\), is a strictly increasing sequence of nonnegative numbers diverging to \( \infty \), the \( p_k \) may be chosen so that \( \sum_{k=3}^{\infty} a_k 2^{-k-2} < \infty \). Choose, for example, each \( p_{k-2} \) large enough so that \( a_k 2^{-k-2} < 1/2^k \).

An increasing sequence of closed sets \( S_n \) having properties (a)-(d) of Lemma 1 will be referred to as a modified Zahorski sequence for \( W \) on \([a, b]\).

For each \( N \) and \( m/2^n \) with \( 0 < m < 2^n \), it is possible, by repeated use of the Lusin-Menchoff Lemma to insert a closed set \( S_{N+m/2^n} \) between \( S_N \) and \( S_{N+1} \setminus \bigcup_{k=1}^{n-1} W_k \) in such a way that a set with smaller subscript is contained in and consists only of points of density of a set with larger subscript. These sets may be chosen so that, in addition, \( |S_{N+1/2^n} \cap (x, x + t)|/|t| > L(1 - 2^{-m+1}) \) for \( x \in S_N \) whenever \( |t| < 1/m \). Details for a similar situation may be found in [1, p. 166]. The essential difference in the present work is that the sets \( S_{N+m/2^n}, m < 2^n \), are all contained in \( S_{N+1} \setminus \bigcup_{k=1}^{n-1} W_k \). Consequently, \( W_{n+1} \setminus \bigcup_{k=1}^{n-1} W_k \subset S_{N+1} \setminus S_{N+m/2^n} \) for each \( N \) and all \( m < 2^n \).

As in [1] for each \( \lambda > 1 \), let \( S_\lambda = \bigcap_{N+m/2^n > \lambda} S_{N+m/2^n} \).

This collection of sets \( \{S_\lambda\}_{\lambda>1} \) will be referred to as a modified Zahorski collection for \( W \) on \([a, b]\).

**Lemma 2.** Let \( W = \bigcup_{n=1}^{\infty} W_n \) be an \( F_a \) set of measure 0 contained in the interval \([a, b]\) and let \( \{a_n\}, n > 1 \), be a strictly increasing sequence of nonnegative numbers diverging to \( \infty \). Let \( \{S_\lambda\}_{\lambda>1} \) be a modified Zahorski collection for \( W \) on \([a, b]\) where the \( p_k \) are chosen so that \( \sum_{k=3}^{\infty} a_k 2^{-k-2} < \infty \).

For each \( x > 1 \), let \( f(x) = a_n + (x - n)(a_{n+1} - a_n) \), where \( n \) is the unique positive integer for which \( n \leq x < n + 1 \). For each \( x \in [a, b] \) let

\[
h(x) = \begin{cases} 
\inf_{\lambda>1} \{\lambda \mid x \in S_\lambda\}, & x \in \bigcup_\lambda S_\lambda, \\
\infty, & x \notin \bigcup_\lambda S_\lambda.
\end{cases}
\]

Define the function \( g \) on \([a, b]\) by
(If \( W \) is also a \( G_\delta \) set, \( \bigcup_\lambda S_\lambda \) may be the entire interval \([a, b]\), in which case the second part of the definition is superfluous.) The function \( g \) has these properties.

(a) \( g \) is bounded below by \( a_1 \) and is identically \( a_1 \) on \( S_1 \).
(b) \( g(w) = a_{N+1} \) for each \( w \in W_{N+1} \setminus \bigcup_{k=1}^N W_k \).
(c) \( g \) is upper-semicontinuous at each \( x \in \bigcup_\lambda S_\lambda \) and \( \lim_{u \to x} g(u) = \infty \) for each \( x \notin \bigcup_\lambda S_\lambda \).

\[
\begin{align*}
g(x) &= \begin{cases} 
f(h(x)) & \text{if } x \in \bigcup_\lambda S_\lambda, \\
\infty & \text{if } x \notin \bigcup_\lambda S_\lambda. 
\end{cases}
\end{align*}
\]

Proof. (a) and (b) follow immediately since \( W_1 \subset S_1 \) and \( W_{N+1} \setminus \bigcup_{k=1}^N W_k \subset S_{N+1} \setminus S_{N+m/2} \) for each \( N \) and \( m < 2^n \).

The function \( g \) is upper-semicontinuous at each \( x \) for which \( g(x) = a_1 \) since \( g \) is bounded below by \( a_1 \). Suppose that \( g(x) > a_1 \) and finite. Then \( h(x) > 1 \). Let \( \epsilon > 0 \) be arbitrary and let \( \delta > 0 \) be such that \( h(x) - \delta > 1 \) and \( f(h(x) - \delta) > f(h(x)) - \epsilon \). Since \( h(x) = \inf_{\lambda>1}(\lambda|x \in S_\lambda) \), \( x \notin S_{h(x)-\delta} \). Let \( t > 0 \) be such that \((x - t, x + t) \cap S_{h(x)-\delta} \) is empty. Such a \( t \) exists since \( S_{h(x)-\delta} \) is closed. For \( u \in (x - t, x + t) \), \( h(u) > h(x) - \delta \) and \( g(u) > f(h(x) - \delta) > f(h(x)) - \epsilon = g(x) - \epsilon \). Thus \( g \) is upper-semicontinuous at \( x \). If \( x \notin \bigcup_\lambda S_\lambda \), then \( h(u) > n \) and \( g(u) > a_n \) for \( u \in (x - t, x + t) \), \( t = \text{dis}(x_1, S_n) \). Thus \( \lim_{u \to x} g(u) = \infty \). Therefore (c) holds.

It remains to verify (d)—a crucial property of \( g \) for future applications. (d) follows immediately from (c) if \( x \notin \bigcup_{\lambda>1} S_\lambda \). Let \( x \in \bigcup_{\lambda>1} S_\lambda \) and let \( n \) be the smallest positive integer for which \( x \in S_n \). For every \( 0 < \epsilon < (n + 1) - h(x) \) and any \( t \),

\[
\left| \frac{1}{t} \int_x^{x+t} g(u) - g(x) \, du \right| < \left| \frac{1}{t} \int_{S_{h(x)}+\epsilon \cap (x,x+t)} g(u) - g(x) \, du \right|
\]

\[
+ \left| \frac{1}{t} \int_{S^n+\epsilon \cap (x,x+t)} g(u) - g(x) \, du \right|
\]

\[
+ \sum_{k=n+2}^\infty \frac{1}{|t|} \int_{S_k \setminus S_{k-1}} g(u) - g(x) \, du.
\]

Since \( h(u) < k \) for \( u \in S_k \setminus S_{k-1} \), \( g(u) < a_k \) for \( u \in S_k \setminus S_{k-1} \). Consequently,
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$$\sum_{k=n+2}^{\infty} \frac{1}{|t|} \int_{S_k \setminus S_{k-1} \cap (x, x + t)} g(u) - g(x) \, du$$

$$\leq \sum_{k=n+2}^{\infty} (a_k - g(x)) \frac{|S_k \setminus S_{k-1} \cap (x, x + t)|}{|t|}.$$

It follows from property (d) of Lemma 1 and from the fact that $x$ is in $S_{k-2}$ for $k > n + 2$, that the right side of this inequality is dominated by

$$\sum_{k=n+2}^{\infty} (a_k - g(x)) \frac{|t|^{-p_k - m + 1}}{|t|} = 2^{-m+1} \sum_{k=n+2}^{\infty} (a_k - g(x))2^{-p_k - 2}$$

when $|t| < 1/m$. This sum approaches 0 as $m \to \infty$ since $p_{k-2}$ and $\sum_{k=n+2}^{\infty} a_k 2^{-p_k - 2} < \infty$.

$$\left| \frac{1}{t} \int_{S_{n+1} \setminus S_{h(x)} + \epsilon \cap (x, x + t)} \ g(u) - g(x) \, du \right|$$

also approaches 0 as $t \to 0$ since this expression is less than or equal to

$$(a_{n+1} - g(x))|S_{n+1} \setminus S_{h(x)} + \epsilon \cap (x, x + t)|/|t|$$

and $x$ is a point of density of both $S_{n+1}$ and $S_{h(x)} + \epsilon$. ($x \in S_{h(x)} + \epsilon/2$ implies that $x$ is a point of density of $S_{h(x)} + \epsilon$).

Finally, consider

$$\left| \frac{1}{t} \int_{S_{h(x)} + \epsilon \cap (x, x + t)} g(u) - g(x) \, du \right|.$$

Let $|t|$ be small enough so that $g(u) \geq g(x) - \epsilon$ for $u \in (x - t, x + t)$. Then,

$$(g(x) - \epsilon) \frac{S_{h(x)} + \epsilon \cap (x, x + t)}{|t|} < \frac{1}{|t|} \int_{S_{h(x)}} g(u) \, du$$

$$< f(h(x) + \epsilon)|S_{h(x)} + \epsilon \cap (x, x + t)|/|t|.$$ 

Consequently,

$$\lim_{t \to 0} \frac{1}{t} \int_{S_{h(x)} + \epsilon \cap (x, x + t)} g(u) - g(x) \, du < \max\{\epsilon, f(h(x) + \epsilon) - g(x)\}.$$ 

Summarizing, we have

$$\lim_{t \to 0} \frac{1}{t} \int_{x + t}^{x + t} g(u) - g(x) \, du < \max\{\epsilon, f(h(x) + \epsilon) - g(x)\}.$$ 

Since $\epsilon$ was arbitrary and $f$ is continuous at $h(x)$, this limit must be 0.

This completes the proof of Lemma 2. Q.E.D.

**Lemma 3.** Let $W = \bigcup_{n=1}^{\infty} W_n$ be an $F_\sigma$ set of measure 0 contained in the open interval $(a, b)$. Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of nonnegative numbers diverging to $\infty$. Let $(S_k)_{k \geq 1}$ be a modified Zahorski collection for $W$ on $[a, b]$ where the $P_k$ are chosen so that $\sum_{k=3}^{\infty} a_k 2^{-p_k - 2} < \infty$. If $f$, $h$ and $g$ are defined as in Lemma 2,
\[ \left| \int_0^R \left( \frac{1}{t^2} \right) \left\{ J \left( (w, w + t) \right) - J \left( (w - t, w) \right) \right\} dt \right| \]

is finite for every \( w \in W \) and positive number \( R \) less than \( \min(b - w, w - a) \). Here \( J \left( (w, w + t) \right) = \int_{(w, w + t)} g(u) \, du \) and \( J \left( (w - t, w) \right) = \int_{(w - t, w)} g(u) \, du \).

**Proof.** The crucial observation here is that for each \( w \in W \setminus \bigcup_{k=0}^{n-1} W_k \), \( n > 1 \), there exists a \( T_w > 0 \) such that \( |S_n \cap (w, w + t)| = 0 \) for all \( t \) with \( |t| < T_w \). This is just (c) of Lemma 1. Consequently, \( n < h(u) < n + 1/2^k \) for almost all \( u \) in the set \( S_{n+1/2^k} (w - t, w + t) \) and \( a_n < g(u) < a_n + (a_{n+1} - a_n)/2^k \) for almost all \( u \) in this set. Apart from the critical role played by this observation, the structure of the proof is very similar to that of Lemma 3 in [1]. Consequently, some details are omitted.

Fix \( R \) and \( w \). For each measurable subset \( M \) of \((0, R)\) define \( J(M) = \int_M g(u) \, du \), \( M(t, +) = M \cap (w, w + t) \), \( M(t, -) = M \cap (w - t, w) \) and \( M_k = S_{k} \setminus S_{k-1} \).

If \( w \in W_1 \), then \( w \in S_1 \) and the result follows from Lemma 3 in [1]. Suppose that \( w \notin W_1 \) and let \( n \) be such that \( w \in W_n \setminus \bigcup_{k=0}^{n-1} W_k \). Set

\[ *= \int_0^R \left( \frac{1}{t^2} \right) \left\{ J \left( (w, w + t) \right) - J \left( (w - t, w) \right) \right\} dt, \]

\[ A = \int_0^R \left( \frac{1}{t^2} \right) \left\{ \sum_{k=n+2}^{\infty} J(M_k(t, +)) - J(M_k(t, -)) \right\} dt, \]

\[ B = \int_0^R \left( \frac{1}{t^2} \right) \left\{ J(S_{n+1}(t, +)) - J(S_{n+1}(t, -)) \right\} dt. \]

Note that \( * = A + B \). Consider \( A \) first. For \( u \in S_k \), \( h(u) < k \) and \( g(u) < a_k \). Thus

\[ |A| < \int_0^R \left( \frac{1}{t^2} \right) \left\{ \sum_{k=n+2}^{\infty} a_k (|M_k(t, +)|) + |M_k(t, -)| \right\} dt. \]

Using (d) of Lemma 1, it can be shown that

\[ |A| \leq 4 \sum_{k=n+2}^{\infty} a_k 2^{-p_{k-2}} \int_0^R \left( \frac{-1/2^{\sqrt{2}}}{} \right) dt, \]

which converges since \( \sum_{k \geq 3} a_k 2^{-p_{k-2}} < \infty \).

Now consider \( B \). To show that \( |B| \) is finite, it suffices to show that the integral converges with \( R \) replaced by \( T_w \), where \( T_w > 0 \) is such that \( |S_n \cap (w, w + t)| = 0 \) for \( |t| < T_w \). Fix \( 0 < t < T_w \) and let \( k \) be the positive integer for which \( 1/2^{k+1} < t^2 < 1/2^k \). Set \( M(t) = (1/t^2) \{ J(S_{n+1}(t, +)) - J(S_{n+1}(t, -)) \} \) and set \( B_s = S_{n+1/t^2} \setminus S_{n+(t-1)/2^s} \), \( s > 1 \). Then
From the observation at the beginning of the proof, it follows that
\[ J(S_{n+1/2t}(t, +)) - J(S_{n+1/2t}(t, -)) \leq a_n + 2^{-k}(a_{n+1} - a_n)|S_{n+1/2t}(t, +)| - a_n|S_{n+1/2t}(t, -)|. \]
The right side of this inequality is in turn dominated by
\[ 2^{-k}(a_{n+1} - a_n)|S_{n+1/2t}(t, -)| + (a_n + 2^{-k}(a_{n+1} - a_n))(|S_{n+1/2t}(t, +)| - |S_{n+1/2t}(t, -)|). \]
The remaining details are essentially the same as those in the proof of Lemma 3 in [1, p. 172]. Q.E.D.

Proof of theorem. If \( e^{ix} \not\in W \), replace \( W_1 \) by \( W_1 \cup \{ e^{ix} \} \). Let \( W_n^* = \{ x \in [-\pi, \pi] \mid e^{ix} \in W_n \} \), and set \( W^* = \bigcup_{n=1}^{\infty} W_n^* \). Set \( a_n = -\ln q_n \). Let \( \{ S_\lambda \}_{\lambda > 1} \) be a modified Zahorski collection for \( W^* \) on \( [-\pi, \pi] \) with the \( p_k \) so chosen that \( \sum_{k=3}^{\infty} a_k 2^{-k-1} < \infty \).

If \( W \) is also a \( G_\delta \) set, let \( \{ S_\lambda \}_{\lambda > 1} \) be such that \( \bigcup_{\lambda>1} S_\lambda = [-\pi, \pi] \). Finally, let \( g \) be the function defined in Lemma 1 and extended periodically to a function on \( C \). Define
\[ H(w) = \exp\left( \frac{-1}{2\pi} \int_{-\pi}^\pi e^{iu + w} g(u) du, \quad w \in D. \right. \]

Since
\[ \lim_{t \to 0} \frac{1}{t} \int_x^{x+t} g(u) du = \begin{cases} g(x), & x \in \bigcup_{\lambda > 1} S_\lambda, \\
\infty, & x \not\in \bigcup_{\lambda > 1} S_\lambda, \end{cases} \]
\[ \lim_{r \to 1} |H(re^{ix})| = \exp(-g(x)) \quad \text{for all } x \in [-\pi, \pi]. \]

Since \( g(x) = -\ln q_1 \) for \( x \in W_1^* \) and \( g(x) = -\ln q_{n+1} \) for \( x \in W_{n+1}^* \setminus \bigcup_{k=1}^{n} W_k^* \), this limit is \( q_1 \) on \( W_1 \) and \( q_{n+1} \) on \( W_{n+1} \setminus \bigcup_{k=1}^{n} W_k^* \). If \( \bigcup_{\lambda>1} S_\lambda = [-\pi, \pi] \), then the limit is clearly never 0.

To show that the radial limit of \( H \) itself exists on \( W_1 \) it suffices to show that \( \lim_{\gamma \to 1} \int_{\gamma}^{\infty} (J((w, w + t)) - J((w - t, w))/t^2 \, dt \) is finite for all \( w \in W \) [1, p. 173]. The finiteness of this integral follows from Lemma 3. Q.E.D.

**REFERENCES**


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