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ON JAMES' QUASI-REFLEXIVE BANACH SPACE

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ABSTRACT. In the James' space J, there exist complemented reflexive subspaces which are not uniformly convexifiable and there are uncountably many mutually nonequivalent unconditional basic sequences in J each of which spans a complemented subspace of J. If $\{y_n\}$ is a block basic sequence with constant coefficients of the unit vector basis of J, then its closed linear span $[y_n]$ is complemented in J and the space $[y_n]$ is either isomorphic to J or to $(\Sigma_{n=1}^{\infty} J_{t_n})_{l_2}$ for some $\{t_n\}$ where $J_n = [e_1, e_2, \ldots, e_n]$.

Introduction. For a sequence $x = (\alpha_1, \alpha_2, \ldots)$ of real numbers let

$$\|x\| = \sup\left\{ \sum_{i=1}^{k} (\alpha_{p_{i+1}} - \alpha_{p_i})^2 \right\}^{1/2} \tag{1}$$

where the sup is over all positive integers k and all increasing sequences $\{p_i\}$ of positive integers. Let J be the Banach space of all x such that $\|x\|$ is finite and $\lim_n \alpha_n = 0$. This remarkable Banach space, with an equivalent norm, was constructed by R. C. James [8], [9] to give an example of a separable Banach space whose canonical embedding in its second conjugate space J^{**} is of codimension one. The study of J, on the one hand, leads to the study of shrinking (resp. boundedly complete) basic sequences in Banach spaces and to the study of Banach spaces which contain subspaces isomorphic to c_0 (resp. l_1). On the other hand, it leads to the study of quasi-reflexive spaces and is a major steppingstone to the study of nonreflexive Banach spaces (e.g., [3]). Recently, James [10], [11] solved two open problems in Banach space structure theory by constructing two ingenious examples of nonreflexive Banach spaces, both of which are closely related to J. Thus a better understanding of the structure of J should aid in understanding the pathology that can occur in a nonreflexive Banach space.

The major known properties of J are proved in [4], [5], [7], [8], [9]. In [2], a general construction of spaces of the same type as J is introduced. Throughout this paper, we shall let N be the set of all positive integers and let $\{e_n\}$ be the unit vector basis of J. For a sequence of elements $\{y_i\}$ in J, we shall let $[y_i]$ denote the closed linear subspace spanned by $\{y_i\}$ in J. For the terminology on bases in Banach spaces, we refer the reader to [13].

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In this paper, we show that there exist complemented reflexive subspaces in J which are not uniformly convexifiable, hence there are complemented reflexive subspaces in J other than l_2. It is well known that J does not have an unconditional basis. However, we show that there are uncountably many mutually nonequivalent unconditional basic sequences in J and every subsymmetric basic sequence in J is equivalent to the unit vector basis of l_2. Finally, although the basis $\{e_n\}$ is conditional, its block basic sequences with constant coefficients behave like those of a symmetric basis; namely, if $\{y_n\}$ is a block basic sequence with constant coefficients of $\{e_n\}$, then $[y_n]$ is complemented in J. Furthermore, $[y_n]$ is either isomorphic to J or to $(\sum_{n=1}^{\infty} J_{t_n})_{l_2}$ for some sequence $\{t_n\}$ in N, where $J_n = [e_1, \ldots, e_n]$, $n = 1, 2, \ldots$.

1. Reflexive subspaces. In this section, we show that there exist complemented reflexive subspaces of J which are not uniformly convexifiable.

DEFINITION. A sequence $\{n_i\}$ in N is called proper if the complement $N \setminus \{n_i\}$ of $\{n_i\}$ in N is infinite.

We prove that for any proper sequence $\{n_i\}$, the space $[e_{n_i}]$ is reflexive and is either isomorphic to l_2 or is not uniformly convexifiable.

THEOREM 1. *For any sequence $\{t_k\}$ in N, the natural basis of $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$ is equivalent to $\{e_{n_i}\}$ where $\{n_i\}$ is the sequence complementary to $\{\sum_{i=1}^{k} t_i + k\}$ in N.*

We first prove a technical lemma concerning the norm of J.

LEMMA 2. *Let $m_1 < n_1 < m_2 < n_2 < \cdots$ be a monotone increasing sequence of positive integers such that $m_{k+1} - n_k \geqslant 2$ for all k. For any real numbers α_i,*

$$\sum_{j=1}^{k} \left\| \sum_{i=m_j}^{n_j} \alpha_i e_i \right\|^2 \leqslant \left\| \sum_{j=1}^{k} \sum_{i=m_j}^{n_j} \alpha_i e_i \right\|^2 \leqslant 2 \sum_{j=1}^{k} \left\| \sum_{i=m_j}^{n_j} \alpha_i e_i \right\|^2. \tag{2}$$

PROOF. The first inequality in (2) is immediate since $\|\sum_{i=m_j}^{n_j} \alpha_i e_i\|$ can be evaluated for each j by using only indices in the interval $[m_j - 1, n_j + 1]$. The second inequality follows from the fact that

$$(a - b)^2 \leqslant 2\left[(a - 0)^2 + (0 - b)^2\right],$$

which can be used to change any (1)-estimate of the middle member of (2) into a sum not larger than the last number.

PROOF OF THEOREM 1. For any sequence $\{t_k\}$ in N, let $m_0 = 0$ and $m_k = \sum_{i=1}^{k} t_i + k$, $k \in N$. Let $\{n_i\}$ be the sequence complementary to $\{m_k\}$ in N. Note that $\{n_i\}$ is a proper sequence. For any element of the form

$$x = \left(\sum_{i=1}^{t_k} \alpha_i^{(1)} e_i, \ldots, \sum_{i=1}^{t_n} \alpha_i^{(n)} e_i, 0, 0, \ldots \right)$$

in $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$, let $y = \sum_{k=1}^{n} \sum_{i=1}^{t_k} \alpha_i^{(k)} e_{m_{k-1}+i}$. By Lemma 2, $\|x\| \leqslant \|y\| \leqslant \sqrt{2}\,\|x\|$. It follows easily that $\{e_{n_i}\}$ is equivalent to the natural basis of $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$.

COROLLARY 3. *If $\{n_i\}$ is a proper sequence then $[e_{n_i}]$ is a reflexive subspace of J.*

PROOF. Let $m_0 = 0$ and let $\{m_k\}$ be the sequence complementary to $\{n_i\}$ in N. If we define $t_k' = m_k - m_{k-1} - 1$, then, after discarding those t_k' that are zero, the same argument as in the proof of Theorem 1 shows that there exists a sequence $\{t_k\}$ in N such that $\{e_{n_i}\}$ is equivalent to the natural basis of $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$. As the latter space is reflexive, the proof is complete.

It is well known that every basis in a uniformly convex space is p-Besselian for some $1 < p < \infty$. On the other hand, if $\limsup_k t_k = \infty$, the natural basis of $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$ is clearly not p-Besselian. Hence, by Corollary 3, the following is immediate.

COROLLARY 4. *For any proper sequence $\{n_i\}$, let $\{m_k\}$ be the sequence complementary to $\{n_i\}$ in N. If*

$$\limsup_k\,(m_{k+1} - m_k) = \infty,$$

then $[e_{n_i}]$ is a reflexive subspace of J which is not uniformly convexifiable.

REMARKS. (i) It is easy to prove that if $\sup_k t_k < \infty$ then $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$ is isomorphic to l_2.

(ii) By a result of Edelstein and Mityagin [4, Lemma 5], for any $t_1 < t_2 < \cdots$ and $s_1 < s_2 < \cdots$, $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$ is isomorphic to $(\sum_{k=1}^{\infty} J_{s_k})_{l_2}$.

THEOREM 5. *For any subsequence $\{e_{n_i}\}$ of $\{e_n\}$, $[e_{n_i}]$ is complemented in J.*

PROOF. We may assume $\{n_i\}$ is a proper sequence. Let $m_0 = 0$ and let $\{m_k\}$ be the sequence complementary to $\{n_i\}$ in N. Let $X = [e_{n_i}]$ and let W be the closed linear space of $A \cup B$, where A is the set of all vectors of the form $\sum_{j=n_r}^{m_s} e_j$ with $n_r - 1 \in \{m_k\}$ and $m_s = \min\{m_k : m_k > n_r\}$, and B is the set of all e_j such that $j, j-1$ are in $\{m_k\}$. Then

$$\|w\| \leqslant \|x + w\| \quad \textit{if } x \in X \textit{ and } w \in W,$$

since $\|w\|$ can be evaluated by using (1) with $\{p_i\} \subset \{m_k\}$ and, for such $\{p_i\}$, the sum in (1) is the same for $x + w$ as for w. Therefore, there exists a projection P of J onto X which satisfies $P^{-1}(0) = W$ and $\|P\| \leqslant 2$.

REMARKS. (i) There exist complemented reflexive subspaces of J which are not uniformly convexifiable.

(ii) In Theorem 5, the subspace W is isometrically isomorphic to J.

(iii) It is well known (e.g., [13, Theorem 16.8]) that a basis $\{e_n\}$ in a Banach space X is unconditional if and only if that for any complementary sequences $\{n_i\}$ and $\{m_i\}$ in N, $X = [e_{n_i}] \oplus [e_{m_i}]$. In a sense, J has the extreme opposite property. By Corollary 3 and Theorem 5, if $\{n_i\}$ is a proper sequence and $\{m_i\}$

is its complement in N, then $[e_{n_i}]$ and $[e_{m_i}]$ are complemented in J, but $J \neq [e_{n_i}] \oplus [e_{m_i}]$.

THEOREM 6. *Let $\{m_k\}$ be a proper sequence. If $\{n_i\}$ is the sequence complementary to $\{m_k\}$ in N and $X = [e_{n_i}]$, then J is isomorphic to the quotient space J/X.*

PROOF. As in the proof of Theorem 5, X has a complement W that is isometric to J. Therefore, J is isomorphic to J/X.

REMARK. By a different method, Theorem 6 was obtained by Edelstein and Mityagin [4, Lemma 6].

2. Unconditional basic sequences. In this section, we show that J has uncountably many mutually nonequivalent unconditional basic sequences but every subsymmetric basic sequence in J is equivalent to the unit vector basis of l_2.

The following proposition is proved by Herman and Whitley [7] and is also an immediate consequence of the proof of Corollary 3.

PROPOSITION 7. *Let*

$$y_n = \sum_{i=p_n}^{q_n} \alpha_i e_i, \qquad n = 1, 2, \ldots,$$

be a bounded block basic sequence of $\{e_n\}$. If $p_{n+1} - q_n > 1$, $n = 1, 2, \ldots$, then $\{y_n\}$ is equivalent to the unit vector basis of l_2.

COROLLARY 8. *J has unique subsymmetric basic sequence up to equivalence.*

PROOF. Let $\{y_n\}$ be a subsymmetric basic sequence in J. Since J does not contain a subspace isomorphic to l_1, we may assume that $\{y_n\}$ is a block basic sequence of $\{e_n\}$ (e.g., [1]). By Proposition 7, $\{y_n\}$ is equivalent to the unit vector basis of l_2.

COROLLARY 9. *Every bounded unconditional block basic sequence $\{y_n\}$ of $\{e_n\}$ is equivalent to the unit vector basis of l_2.*

PROOF. This follows immediately from Proposition 7 and the fact that, since $\{y_n\}$ is an unconditional basic sequence, $\sum_{n=1}^{\infty} \alpha_n y_n$ converges if and only if $\sum_{n=1}^{\infty} \alpha_{2n} y_{2n}$ and $\sum_{n=1}^{\infty} \alpha_{2n-1} y_{2n-1}$ converge.

THEOREM 10. *Let $y_n = \sum_{i=p_n}^{q_n} \alpha_i e_i$, $n = 1, 2, \ldots$, be a normalized block basic sequence of $\{e_n\}$. If $p_{n+1} - q_n > 1$, $n = 1, 2, \ldots$, then $[y_n]$ is complemented in J.*

PROOF. Let $\{n_i\}$ denote the sequence $p_1, \ldots, q_1, p_2, \ldots, q_2, \ldots$. By Corollary 3, there exists a sequence $\{t_k\}$ in N such that $\{e_{n_i}\}$ is equivalent to the natural basis of $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$. Let T be the isomorphism establishing the equivalence. For each $k \in N$, choose $f_k \in J_{t_k}^*$, $\|f_k\| = 1$, such that $f_k(Ty_k) = \|Ty_k\|$. For $x = \sum_{i=1}^{\infty} \alpha_{n_i} e_{n_i} \in [e_{n_i}]$, define

$$P(x) = T^{-1}\left\{\sum_{k=1}^{\infty} \frac{1}{\|Ty_k\|} f_k \left[T\left(\sum_{j=p_k}^{q_k} \alpha_j e_j\right)\right] Ty_k\right\}.$$

From Lemma 2 and the definition of the norm in $(\Sigma_{k=1}^{\infty} J_{t_k})_{l_2}$,

$$\|P(x)\| \leqslant \|T\|\,\|T^{-1}\|\left\{\sum_{k=1}^{\infty}\left\|\sum_{j=p_k}^{q_k} \alpha_j e_j\right\|^2\right\}^{1/2} < \sqrt{2}\,\|x\|.$$

It follows that P is well defined. Routine calculations show that P is a bounded projection of $[e_{n_i}]$ onto $[y_n]$. Since $[e_{n_i}]$ is complemented in J by Theorem 5, the proof is complete.

Corollary 11. *For every infinite dimensional subspace E in J there exists a subspace F in E such that F is complemented in J and F is isomorphic to l_2.*

Proof. E contains a normalized basic sequence $\{z_n\}$ that is equivalent to the unit vector basis of l_2 (e.g., [7]). We may choose a normalized block basis $\{y_n\}$ of $\{e_n\}$, which satisfies the condition of Theorem 10, and a subsequence $\{w_n\}$ of $\{z_n\}$ such that

$$\sum_{n=1}^{\infty} \|y_n - w_n\| < \sqrt{2}/2.$$

By Theorem 10, there exists a projection P from J onto $[y_n]$ with $\|P\| < \sqrt{2}$. If $\{y_n^*\}$ are the coefficient functionals of $\{y_n\}$, then $\|y_n^*\| = 1$ for all n. Since

$$\|P\| \sum_{n=1}^{\infty} \|y_n - w_n\| \cdot \|y_n^*\| < 1,$$

by a result of C. Bessaga and A. Pelczynski (e.g., [13]), we conclude that $[w_n]$ is complemented in J.

The results of Theorems 6 and 10 also suggest that J might be a primary space. Recall that a Banach space X is primary if for any projection P on X either PX or $(I - P)X$ is isomorphic to X.

Next, we use the fact [5] that c_0 is finitely representable in J to show that J has uncountably many mutually nonequivalent unconditional basic sequences. Let l_∞^n (resp. l_1^n) be the Banach space of all $x = (a_1, \ldots, a_n)$ with

$$\|x\| = \sup_i |a_i| \quad \left(\text{resp. } \|x\| = \sum_{i=1}^{n} |a_i|\right).$$

The following result is an immediate consequence of [5].

Proposition 12. *Given $\lambda > 1$, then for any positive integer n there exists an integer m such that there exists an isomorphism T from l_∞^n into J_m such that $\lambda^{-1}\|x\| \leqslant \|Tx\| \leqslant \lambda\|x\|$ for all $x \in l_\infty^n$.*

Theorem 13. *Let $\{n_i\}$ be a sequence of positive integers. Then*

(i) *The Banach space $(\Sigma_{i=1}^{\infty} l_\infty^{n_i})_{l_2}$ is isomorphic to a complemented subspace of J,*

(ii) *The Banach space* $(\sum_{i=1}^{\infty} l_1^{n_i})_{l_2}$ *is isomorphic to a subspace of* J.

PROOF. (i) By Proposition 12, there exist a sequence $\{m_i\}$ and isomorphisms T_i from $l_\infty^{n_i}$ into J_{m_i} with $\sup_i \|T_i\| \leqslant \sqrt{2}$ and $\sup \|T_i^{-1}\| \leqslant \sqrt{2}$. Hence $(\sum_{i=1}^{\infty} l_\infty^{n_i})_{l_2}$ is isomorphic to a subspace of $(\sum_{i=1}^{\infty} J_{m_i})_{l_2}$. Furthermore, for each $i = 1, 2, \ldots$, there exists a projection P_i from J_{m_i} onto $T_i(l_\infty^{n_i})$ with $\|P_i\| \leqslant 2$ and by Theorem 5, $(\sum_{i=1}^{\infty} J_{m_i})_{l_2}$ is isomorphic to a complemented subspace of J. Thus $(\sum_{i=1}^{\infty} l_i^{n_i})_{l_2}$ is isomorphic to a complemented subspace of J.

(ii) This follows immediately from (i) and the fact that l_1 is finitely representable in c_0.

COROLLARY 14. *There are uncountably many mutually nonequivalent unconditional basic sequences in* J.

3. Block basic sequences of $\{e_n\}$ with constant coefficients. It is well known that if $\{x_n\}$ is a symmetric basis of a Banach space X, then for every block basic sequence $\{y_n\}$, the "averaging projection" is a bounded projection from X onto $[y_n]$. Although $\{e_n\}$ is conditional, we show in this section, by making use of the "averaging projection", that every block basic sequence of $\{e_n\}$ with constant coefficients spans a complemented subspace in J. The only other known conditional bases in Banach spaces with this property seem to be the conditional bases in l_2.

The following simple lemma follows immediately from the definition of the norm in J.

LEMMA 15. *For any* $p_1 < p_2 < \cdots < p_{n+1}$, *let*

$$y_i = \sum_{k=p_i+1}^{p_{i+1}} e_k, \qquad i = 1, 2, \ldots, n.$$

Then

$$\left\| \sum_{i=1}^{n} \alpha_i e_i \right\| \leqslant \left\| \sum_{i=1}^{n} \alpha_i y_i \right\| \leqslant \sqrt{2} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|$$

for all real numbers $\alpha_1, \ldots, \alpha_n$.

THEOREM 16. *Suppose* $p_1 \leqslant q_1 < p_2 \leqslant q_2 < \cdots$. *Let* $y_n = \sum_{i=p_n}^{q_n} e_i$, $n = 1, 2, \ldots$. *Then*

(i) *If* $p_{n+1} = q_n + 1$ *for all except finitely many* n, $[y_n]$ *is isomorphic to* J.

(ii) *If* $p_{n+1} > q_n + 1$ *for infinitely many* n, *there exists a sequence of positive integers* $\{t_k\}$ *such that* $[y_n]$ *is isomorphic to* $(\sum_{k=1}^{\infty} J_{t_k})_{l_2}$.

(iii) $[y_n]$ *is complemented in* J.

PROOF. (i) If there exists $m \in N$ such that $p_{n+1} = q_n + 1$ for all $n \geqslant m$, then $[y_n]$ is isomorphic to J by Lemma 15.

(ii) Let $\{n_k\}$ be the sequence of all positive integers n such that $p_{n+1} > q_n + 1$, and let $n_0 = 0$. Let $t_k = n_k - n_{k-1}$, $k = 1, 2, \ldots$. For each fixed k, by Lemma 15, there exists an isomorphism T_k from J_{t_k} onto $[y_{n_{k-1}+1},$

$y_{n_{k-1}+2}, \ldots, y_{n_k}]$ such that $\|x\| \leq \|T_k x\| \leq 2\|x\|$ for all $x \in J_{t_k}$. Using the same argument as the proof of Theorem 1, we conclude that $[y_n]$ is isomorphic to $(\Sigma_{k=1}^{\infty} J_{t_k})_{l_2}$.

(iii) As in the proof of Theorem 5, there is a norm one projection P_1 of J onto the closed linear span W of

$$[y_n] \cup \{e_i: q_k < i < p_{k+1} \text{ for some } k\}.$$

Since W is isometric to J, via an isometry which maps the e_i's of J onto the y_n's and e_i's $(q_k < i < p_{k+1})$ of W, it follows by the proof of Theorem 5, that there is a projection P_2 of W onto $[y_n]$ with $\|P_2\| \leq 2$. Thus there is a projection P of J onto $[y_n]$ with $\|P\| \leq 2$.

REMARK. By a result of Edelstein and Mityagin [**4**, Lemma 5], if $\limsup_k t_k = \infty$, then $(\Sigma_{k=1}^{\infty} J_{t_k})_{l_2}$ is isomorphic to $(\Sigma_{n=1}^{\infty} J_n)_{l_2}$.

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