ESSENTIALLY HERMITIAN OPERATORS ON $l_1$
ARE COMPACT PERTURBATIONS OF HERMITIANS

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Abstract. In this paper, we present a solution to one case of a problem of F. F. Bonsall; namely, that every essentially Hermitian operator on $l_1$ is a compact perturbation of a Hermitian operator.

1. Introduction. Let $B(X)$ and $C(X)$ denote, respectively, the algebras of bounded and compact linear operators on a Banach space $X$. The quotient algebra $B(X)/C(X)$, called the Calkin algebra, has been the object of much study lately, especially for the case when $X$ is a complex separable Hilbert space $H$. Of primary interest is the problem of which properties of a residue class modulo $C(X)$ can actually be exhibited by a representative of that class. For example, it is well known that for each $T \in B(H)$, there is a $K \in C(H)$ such that $\|T + K\| = \|T\|$. In [6] Stampfli shows that for each $T \in B(H)$ there is a $K \in C(H)$ such that the spectrum of $T + K$ is the same as the essential spectrum of $T$. In [5] it is shown that if $X = l_p$, $1 < p < \infty$, and if the essential numerical range of $T \in B(X)$ contains interior points, then there is a $K \in C(X)$ such that the numerical range of $T + K$ is the same as the essential numerical range of $T$. In case $X = H$, a Hilbert space, the condition that the essential numerical range contain interior points can be dropped. The present paper arose out of an attempt to establish the same result for $X = l_p$, $p \neq 2$. It can be dropped in Hilbert space because every essentially Hermitian operator can be written as a compact perturbation of a Hermitian operator. However, in [1], F. F. Bonsall asks if every essentially Hermitian operator in $B(l_p)$, $p \neq 2$, is a compact perturbation of a Hermitian operator. Hence we must work on this problem first. In this paper, we solve one case of Bonsall’s question by showing that every essentially Hermitian operator on $l_1$ is a compact perturbation of a Hermitian operator.

2. Definitions and notation. The notation is that used by Bonsall and Duncan [2], [3]. Let $\Pi = \{(x, f) \in X \times X^*: \|x\| = \|f\| = 1, f(x) = 1\}$. The spatial numerical range of $T \in B(X)$ is the set $V(T) = \{f(Tx): (x, f) \in \Pi\}$.

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The algebra numerical range of $T$ is the set $V(B(X), T) = \{s(T): s$ is a normalized state in $B(X)^*\}$. The essential numerical range of $T$ is the algebra numerical range of the residue class $T + C(X)$ in the Calkin algebra $B(X)/C(X)$. It can be shown that $V(B(X), T)$ is the closed convex hull of $V(T)$. Hence $V(T) \subseteq R$ if and only if $V(B(X), T) \subseteq R$. Such operators are called Hermitian. If the essential numerical range, denoted by $\text{Vess}(T)$, is contained in $R$, $T$ is called essentially Hermitian. If $T$ is a compact perturbation of a Hermitian operator, then $T$ is called almost Hermitian.

3. Preliminary facts. A linear operator $T$ on $l_1$ can be represented by an infinite matrix $\{a_{jk}\}$, $j, k = 1, 2, \ldots$. $T$ is bounded if and only if $\sup_{j \geq 1} \sum_{k=1}^{\infty} |a_{jk}| < \infty$ and $T$ is compact if and only if $\lim_{n \to \infty} \sup_{j \leq n} \sum_{k=1}^{n} |a_{jk}| = 0$. See [7, p. 278].

Following [1], let $R(T)$ be the operator defined by the matrix $\{b_{jk}\}$, where $b_{jk} = (\text{Re } a_{jk}) \delta_{jk}$. Then $T$ is almost Hermitian if and only if $T - R(T)$ is compact.

A complex number $z$ belongs to $\text{Vess}(T)$ if and only if there is a state $s$ in $B(X)^*$ which annihilates $C(X)$ such that $s(T) = z$. See [3, p. 127].

4. Theorem. If $T \in B(l_1)$ is essentially Hermitian, then $T$ is almost Hermitian.

Proof. Let $\text{glim}$ denote a generalized Banach limit as described in [4, p. 856]. If $(x_n, f_n)$ is a sequence in $\Pi$ such that the first $n$ coordinates of $x_n$ and $f_n$ are 0, then $\text{glim}_{n \to \infty} \langle f_n, Cx_n \rangle = \lim_{n \to \infty} \langle f_n, Cx_n \rangle = 0$ for each compact operator $C$. It follows that for an arbitrary operator $T$, $\text{glim}_{n \to \infty} \langle f_n, Tx_n \rangle$ is a number in $\text{Vess}(T)$.

Now suppose $T$ is not almost Hermitian. Then $T - R(T)$ is not compact. But $T - R(T)$ agrees with $T$ except on the diagonal, and the diagonal of $T - R(T)$ is pure imaginary. By the noncompactness of $T - R(T)$, we therefore have $\lim_{n \to \infty} \sup_{j \leq n} \sum_{k=1}^{\infty} |a_{jk}| > 0$. But since $T - R(T)$ is bounded, we also have $\lim_{n \to \infty} \sum_{j=n+1}^{\infty} |a_{jk}| = 0$ for each $k$. It follows that there exists an $\varepsilon > 0$ such that for every $n$, there is a $k_n > n$ such that $\sum_{j=n+1}^{\infty} |a_{jk_n}| + |\text{Im } a_{k_n,k_n}| > \varepsilon$. If possible, always choose $k_n$ such that $|\text{Im } a_{k_n,k_n}| > 0$. If this is not possible, then consider $-T$ rather than $T$. Hence, without loss of generality, we assume $|\text{Im } a_{k_n,k_n}| > 0$.

Now let $x_n$ be the sequence whose $k_n$ coordinate is $i$ (the imaginary unit), and whose remaining coordinates are 0. Let $\theta_{j_n} = \arg a_{j_n,k_n}$, and let $f_n$ be the sequence whose first $n$ coordinates are 0, whose $k_n$ coordinate is $-i$, and whose remaining coordinates in order are $\exp(-i \theta_{j_n}), j = n + 1, \ldots, j \neq k_n$. Then $(x_n, f_n) \in \Pi$ for each $n$, and

$$\langle f_n, (T - R(T))x_n \rangle = \left( \sum_{j=n}^{\infty} |a_{jk_n}| + |\text{Im } a_{k_n,k_n}| \right) i.$$

Hence $\lim_{n \to \infty} \langle f_n, (T - R(T))x_n \rangle$ is a pure imaginary number with
imaginary part exceeding $\varepsilon$. Hence $T - R(T)$ is not essentially Hermitian. Since $R(T)$ is Hermitian, it follows that $T$ is not essentially Hermitian, which is a contradiction.

REFERENCES


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