AFFINE COMPLETE ORTHOLATTICES

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ABSTRACT. Every finite orthomodular lattice is affine complete.

In universal algebra the concept of the polynomial ring has been general-
ized to the concept of polynomial algebra \( A[x_1, \ldots, x_n] \) of a given algebra \( A \)
in a variety [4]. By every polynomial \( \psi(x_1, \ldots, x_n) \in A[x_1, \ldots, x_n] \) one can
define a function \( \psi: A^n \to A \) such that every element \( (a_1, \ldots, a_n) \in A^n \)
is mapped to \( \psi(a_1, \ldots, a_n) \). These functions are called polynomial functions of
\( A \). Similarly, as in the variety of rings, the question of which functions \( f: A^n \to A \)
are polynomial functions of \( A \) is of major interest.

The function \( f: A^n \to A \) is called compatible if for every congruence
relation \( \theta \) on the algebra \( A \), \( a_1 \theta b_1, \ldots, a_n \theta b_n, a_1, \ldots, a_n, b_1, \ldots, b_n \in A \)
imply \( f(a_1, \ldots, a_n) \theta f(a_1, \ldots, a_n) \). Let us notice that every polynomial
function is compatible [4]. The algebra \( A \) is called polynomially complete if
every function of \( A \) is a polynomial function of \( A \). In this case \( A \) is simple
and finite [4]. The algebra \( A \) is called affine complete if every compatible
function is a polynomial function of \( A \) [6], [8]. A lot of examples can be found
in [8] where it is also mentioned that finite Boolean algebras are affine
complete. This result is essentially due to Grätzer [3] and it is our purpose to
extend it to orthomodular lattices. In order to study polynomial functions on
ortholattices we shall do it within a more general concept.

DEFINITION. The algebra \( \mathcal{L} = (L; \wedge, \vee, \cdot) \) is called a polarity lattice if \( \mathcal{L} \) is
a lattice concerning the operations \( \wedge, \vee \) and if the unary operation \( \cdot \) has the
properties

1. \( (x')' = x \),
2. \( (x \vee y)' = x' \wedge y' \).

It is clear that every ortholattice and every Boolean algebra are polarity
lattices. We also have to use some results on order-polynomial-complete
lattices [7]. The lattice \( V \) is order-polynomial-complete if every order pre-
serving function is a polynomial function of \( V \). We call the polarity lattice
\( \mathcal{L} = (L; \wedge, \vee, \cdot) \) order-polynomial-complete if the underlying lattice \( V =
(L; \wedge, \vee) \) is order-polynomial-complete.

LEMMA. If the finite polarity lattice \( L \) is order-polynomial-complete then \( L \) is
polynomially complete.

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Proof. Let \( \{0, \ldots, a, \ldots, 1\} \) be the set of elements of \( L \) and let us consider \( f: L \to L \) with the decomposition of \( f \) by

\[
f(x) = f_0(x) \lor \cdots \lor f_a(x) \lor \cdots \lor f_1(x)
\]

where

\[
f_a(x) = \begin{cases} f(a) & \text{if } x = a, \\ 0 & \text{otherwise}. \end{cases}
\]

For a further decomposition consider

\[
f_a(x) = \psi_a(x) \land \rho_a(x)
\]

where

\[
\psi_a(x) = \begin{cases} f(a) & \text{if } x > a, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\rho_a(x) = \begin{cases} f(a) & \text{if } x < a, \\ 0 & \text{otherwise}. \end{cases}
\]

It is easy to see that \( \psi_a \) is an order preserving function. By hypothesis \( \psi_a \) is a polynomial function. Now consider

\[
n_a(x) = \begin{cases} f(a) & \text{if } x > a', \\ 0 & \text{otherwise}, \end{cases}
\]

which is also order preserving and therefore a polynomial function. If we substitute \( x' \) for the variable \( x \) in the polynomial function \( n_a \), we get a polynomial function of the polarity lattice \( L \) with \( n_a(x') = \rho_a(x) \). Therefore \( f \) is a polynomial function of \( L \). If the unary functions of \( L \) are polynomial functions, then the binary functions are polynomial functions, too. Consider \( h: L^2 \to L \) and decompose

\[
h(x, y) = h_{00}(x, y) \lor \cdots \lor h_{ab}(x, y) \lor \cdots \lor h_{11}(x, y),
\]

where

\[
h_{ab}(x, y) = \begin{cases} h(a, b) & \text{if } x = a \text{ and } y = b, \\ 0 & \text{otherwise}. \end{cases}
\]

We have the further decomposition \( h_{ab}(x, y) = \psi_{ab}(x) \land \rho_{ab}(y) \) where

\[
\psi_{ab}(x) = \begin{cases} f(a, b) & \text{if } x = a, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\rho_{ab}(y) = \begin{cases} f(a, b) & \text{if } y = b, \\ 0 & \text{otherwise}. \end{cases}
\]

By Theorem 11.2 in [4] \( L \) is polynomially complete.

Theorem 1. The finite simple orthomodular lattice \( L \) is polynomially complete.

Proof. It is well known that \( L \) is relatively complemented [1, p. 53]. As \( L \) is finite, Corollary 2 of [1, p. 53] implies that every element of \( L \setminus \{0\} \) is the finite join of atoms of \( L \). Therefore \( L \) is atomistic and relatively complemented. The hypothesis for Theorem 10.14 of [5, p. 47] is fulfilled. As \( L \) is simple it is also irreducible. By 10.14.3 of [5] the atoms of \( L \) are projective to each other. That implies that the atoms of \( L \) are pseudoprojective to each other in the sense of Definition 3 of [7]. Now by Satz 3 of [7] \( L \) is
order-polynomial-complete, and by the lemma \( L \) is polynomially complete.

**Theorem 2.** Every finite orthomodular lattice \( L \) is affine complete.

**Proof.** If \( L \) is a finite orthomodular lattice then \( L \) is the direct product of simple orthomodular lattices which are polynomially complete. By [8, 4.9 Korollar] this is a diagonal product, and by [8, 5.6 Korollar] \( L \) is affine complete.

**References**


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