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AN UNBAIREABLE STRATIFIABLE SPACE

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ABSTRACT. There is a stratifiable space which cannot be densely embedded in a stratifiable Baire space, in fact not even in a regular Baire σ-space. Every regular Baire σ-space, hence every stratifiable Baire space, has a dense metrizable G_δ-subspace.

1. Introduction. If $\mathfrak{M}$ is a class of generalized metrizable spaces, there is considerable interest in the following questions about $\mathfrak{M}$:

(I) Does every Baire space in $\mathfrak{M}$ have a dense metrizable (preferably G_δ-) subspace?

(II) Can every space in $\mathfrak{M}$ be densely embedded in a Baire space in $\mathfrak{M}$?

In this note we answer these questions for the class of stratifiable spaces and for the class of σ-spaces.

PROPOSITION. *Every Baire σ-space, hence every stratifiable Baire space, has a dense metrizable G_δ-subspace.*

EXAMPLE. *There is a stratifiable space which cannot be densely embedded in a stratifiable Baire space, and in fact not even in a regular Baire σ-space.*

The "hence" and the "not even" are justified by Heath's theorem that every stratifiable space is a σ-space **[HH]**. The key to the Example is the following observation; π and c are the *π-weight* and the *cellularity*, respectively; see §2.

LEMMA. *Let $\mathfrak{M}$ be a class of regular spaces such that*

(1) *every Baire space in $\mathfrak{M}$ has a dense metrizable subspace*; *and*

(2) *every space in $\mathfrak{M}$ can be densely embedded in a Baire space in $\mathfrak{M}$.*

Then $\pi(X) = c(X)$ for every $X \in \mathfrak{M}$.

2. Conventions and definitions. All spaces are T_1. Cardinals are initial (von Neumann) ordinals, κ always denotes a cardinal, ω is ω_0.

A family $\mathcal{Q}$ of subsets of a space X is called a *net* if for every open $U \subseteq X$ and every $x \in U$ there is an $A \in \mathcal{Q}$ with $x \in A \subseteq U$. A space will be called a *σ-space* if it has a σ-discrete net consisting of closed sets. (Some authors do not add the restriction "consisting of closed sets," e.g. **[O_2]**. For regular spaces this is irrelevant, of course, and stratifiable spaces are regular.)

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Stratifiable spaces are defined in **[B₁]**.

A family $\mathfrak{B}$ of subsets of a space X is called a *π-base* if every member of $\mathfrak{B}$ is a nonempty open set, and if every nonempty open set of X includes a member of $\mathfrak{B}$. The cardinal functions $\pi(X)$, the *π-weight* of X, and $c(X)$, the *cellularity* of X, are defined by

$\pi(X) = \min\{\kappa\colon X$ has a π-base of cardinality $\kappa\}$;

$c(X) = \sup\{\kappa\colon$ there is a disjoint open family in X with cardinality $\kappa\}$.

3. Proof of the Proposition. Let X be a Baire σ-space. Let $\mathcal{Q} = \bigcup_{n\in\omega}\mathcal{Q}_n$ be a net for X consisting of closed sets, with each $\mathcal{Q}_n$ discrete. For each $A \in \mathcal{Q}$ the boundary Bd A is nowhere dense. So for each $n \in \omega$ the set

$$B_n = \bigcup\{\operatorname{Bd} A\colon A \in \mathcal{Q}_n\}$$

is nowhere dense and closed, since $\mathcal{Q}_n$ is discrete. So $M = X - \bigcup_{n\in\omega}B_n$ is a dense G_δ-subspace of X. Clearly the family $\{M \cap A\colon A \in \mathcal{Q}\}$ is a σ-discrete net consisting of open and closed sets for the subspace M, hence M is regular and has a σ-discrete base. So M is metrizable.

4. Proof of the Lemma. It is well known that if S is a dense subspace of the space T, then $c(S) = c(T)$, and if T is regular then also $\pi(S) = \pi(T)$, cf. **[J]**. So if $X \in \mathfrak{M}$ is a dense subspace of $Y \in \mathfrak{M}$ which has a dense metrizable subspace M, then

$$c(X) = c(Y) = c(M) = \pi(M) = \pi(Y) = \pi(X).$$

5. Construction of the Example. Since $c(X) \leqslant |X|$ for every X, it follows from the Proposition and the Lemma and Heath's theorem, quoted in the introduction, that we only have to construct a stratifiable space X with $\pi(X) > |X|$.

If $\langle X_\alpha\rangle_{\alpha\in\kappa}$ is a family of spaces, we denote the *box product* of this family by $\square_{\alpha\in\kappa}X_\alpha$. Given $p \in \square_{\alpha\in\kappa}X_\alpha$ we define a subspace Ξ_p by

$$\Xi_p = \left\{x \in \mathop{\square}_{\alpha\in\kappa} X_\alpha\colon x_\alpha = p_\alpha \text{ for all but finite many } \alpha\text{'s}\right\}$$

[vD]. It is known that Ξ_p is stratifiable for any p, if all X_α's are metrizable **[vD]**, or just stratifiable **[B₂]**. In our example we let $\kappa = \omega$, let the X_α's be $\mathbf{Q}$, the rationals, and let p be arbitrary. Clearly $|\Xi_p| = \omega$, so it suffices to show that $\pi(\Xi_p) > \omega$. That we prove by a straightforward diagonalization argument. To this end we need the following

FACT. *If B is a nonempty open set in Ξ_p, then for each $k \in \omega$ there is a $b \in B$ with $b_k \neq p_k$.*

Indeed, there is a sequence $\langle U_n\rangle_{n\in\omega}$ of open sets in $\mathbf{Q}$ with $\varnothing \neq \Xi_p \cap \prod_{n\in\omega}U_n \subseteq B$. Since $|U_k| \geqslant 2$, we can find $b \in \Xi_p \cap \prod_{n\in\omega}U_n$ with $b_k \neq p_k$. Let $\mathfrak{B}$ be any countable family of nonempty open sets in Ξ_p. Enumerate $\mathfrak{B}$ as $\langle B_k\rangle_{k\in\omega}$. For each $k \in \omega$ choose $c(k) \in B_k$ with $c(k)_k \neq p_k$. Then

$$U = \Xi_p \cap \prod_{k\in\omega}\left(\mathbf{Q} - \{c(k)_k\}\right)$$

is a nonempty (for $p \in U$) open set in Ξ_p which does not include any member of $\mathfrak{B}$ (for $c(k) \notin U$ for all $k \in \omega$). Hence $\mathfrak{B}$ is not a π-base.

6. Completeness. By a theorem of Borges [$\mathbf{B_1}$] and, independently, Okuyama [$\mathbf{O_1}$], a space is metrizable iff it is a paracompact p-space with a G_δ-diagonal, in particular, iff it is a stratifiable p-space. So every nonmetrizable stratifiable space is uncompletable in the sense that it cannot be embedded in a stratifiable Čech-complete space.

On the other hand there is a nonmetrizable stratifiable space X which has a base $\mathfrak{B}$, consisting of clopen sets, such that every centered subfamily of $\mathfrak{B}$ has nonempty intersection (see [**AL**] for a survey of completeness properties of this sort): Let X be the subspace $\mathbf{N} \cup \{p\}$ of $\beta\mathbf{N}$ for some $p \in \beta\mathbf{N}$. (The subspace $P \cup Q$ of [$\mathbf{vD_2}$, 2.3] would be a first countable such space.)

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