AN UNBAIREABLE STRATIFIABLE SPACE

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ABSTRACT. There is a stratifiable space which cannot be densely embedded in a stratifiable Baire space, in fact not even in a regular Baire $\sigma$-space. Every regular Baire $\sigma$-space, hence every stratifiable Baire space, has a dense metrizable $G_\delta$-subspace.

1. Introduction. If $\mathcal{M}$ is a class of generalized metrizable spaces, there is considerable interest in the following questions about $\mathcal{M}$:

(I) Does every Baire space in $\mathcal{M}$ have a dense metrizable (preferably $G_\delta$-) subspace?

(II) Can every space in $\mathcal{M}$ be densely embedded in a Baire space in $\mathcal{M}$?

In this note we answer these questions for the class of stratifiable spaces and for the class of $\sigma$-spaces.

PROPOSITION. Every Baire $\sigma$-space, hence every stratifiable Baire space, has a dense metrizable $G_\delta$-subspace.

EXAMPLE. There is a stratifiable space which cannot be densely embedded in a stratifiable Baire space, and in fact not even in a regular Baire $\sigma$-space.

The “hence” and the “not even” are justified by Heath’s theorem that every stratifiable space is a $\sigma$-space [HHJ]. The key to the Example is the following observation; $\pi$ and $c$ are the $\pi$-weight and the cellularity, respectively; see §2.

LEMMA. Let $\mathcal{M}$ be a class of regular spaces such that

(1) every Baire space in $\mathcal{M}$ has a dense metrizable subspace; and

(2) every space in $\mathcal{M}$ can be densely embedded in a Baire space in $\mathcal{M}$.

Then $\pi(X) = c(X)$ for every $X \in \mathcal{M}$.

2. Conventions and definitions. All spaces are $T_1$. Cardinals are initial (von Neumann) ordinals, $\kappa$ always denotes a cardinal, $\omega$ is $\omega_0$.

A family $\mathcal{B}$ of subsets of a space $X$ is called a net if for every open $U \subseteq X$ and every $x \in U$ there is an $A \in \mathcal{B}$ with $x \in A \subset U$. A space will be called a $\sigma$-space if it has a $\sigma$-discrete net consisting of closed sets. (Some authors do not add the restriction “consisting of closed sets,” e.g. $[O_2]$. For regular spaces this is irrelevant, of course, and stratifiable spaces are regular.)

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Stratifiable spaces are defined in [B_1].

A family \( \mathcal{B} \) of subsets of a space \( X \) is called a \( \pi \)-base if every member of \( \mathcal{B} \) is a nonempty open set, and if every nonempty open set of \( X \) includes a member of \( \mathcal{B} \). The cardinal functions \( \pi(X) \), the \( \pi \)-weight of \( X \), and \( c(X) \), the cellularity of \( X \), are defined by

\[
\pi(X) = \min\{\kappa : X \text{ has a } \pi \text{-base of cardinality } \kappa\};
\]

\[
c(X) = \sup\{\kappa : \text{there is a disjoint open family in } X \text{ with cardinality } \kappa\}.
\]

3. Proof of the Proposition. Let \( X \) be a Baire \( \sigma \)-space. Let \( \mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n \) be a net for \( X \) consisting of closed sets, with each \( \mathcal{A}_n \) discrete. For each \( A \in \mathcal{A} \) the boundary \( \text{Bd} A \) is nowhere dense. So for each \( n \in \omega \) the set

\[
B_n = \bigcup \{\text{Bd} A : A \in \mathcal{A}_n\}
\]

is nowhere dense and closed, since \( \mathcal{A}_n \) is discrete. So \( M = X - \bigcup_{n \in \omega} B_n \) is a dense \( G_\delta \)-subspace of \( X \). Clearly the family \( \{M \cap A : A \in \mathcal{A}\} \) is a \( \sigma \)-discrete net consisting of open and closed sets for the subspace \( M \), hence \( M \) is regular and has a \( \sigma \)-discrete base. So \( M \) is metrizable.

4. Proof of the Lemma. It is well known that if \( S \) is a dense subspace of the space \( T \), then \( c(S) = c(T) \), and if \( T \) is regular then also \( \pi(S) = \pi(T) \), cf. [J].

So if \( X \in \mathcal{M} \) is a dense subspace of \( Y \in \mathcal{M} \) which has a dense metrizable subspace \( M \), then

\[
c(X) = c(Y) = c(M) = \pi(M) = \pi(Y) = \pi(X).
\]

5. Construction of the Example. Since \( c(X) \leq |X| \) for every \( X \), it follows from the Proposition and the Lemma and Heath’s theorem, quoted in the introduction, that we only have to construct a stratifiable space \( X \) with \( \pi(X) > |X| \).

If \( \langle X_\alpha \rangle_{\alpha \in \kappa} \) is a family of spaces, we denote the box product of this family by \( \prod_{\alpha \in \kappa} X_\alpha \). Given \( p \in \prod_{\alpha \in \kappa} X_\alpha \) we define a subspace \( \Xi_p \) by

\[
\Xi_p = \{x \in \prod_{\alpha \in \kappa} X_\alpha : x_\alpha = p_\alpha \text{ for all but finite many } \alpha \text{'s}\}
\]

[vD]. It is known that \( \Xi_p \) is stratifiable for any \( p \), if all \( X_\alpha \)'s are metrizable [vD], or just stratifiable [B_2]. In our example we let \( \kappa = \omega \), let the \( X_\alpha \)'s be \( \mathbb{Q} \), the rationals, and let \( p \) be arbitrary. Clearly \( |\Xi_p| = \omega \), so it suffices to show that \( \pi(\Xi_p) > \omega \). That we prove by a straightforward diagonalization argument. To this end we need the following

**Fact.** If \( B \) is a nonempty open set in \( \Xi_p \), then for each \( k \in \omega \) there is a \( b \in B \) with \( b_k \neq p_k \).

Indeed, there is a sequence \( \langle U_n \rangle_{n \in \omega} \) of open sets in \( \mathbb{Q} \) with \( \emptyset \neq \Xi_p \cap \Pi_{n \in \omega} U_n \subseteq B \). Since \( |U_k| > 2 \), we can find \( b \in \Xi_p \cap \Pi_{n \in \omega} U_n \) with \( b_k \neq p_k \). Let \( \mathcal{B} \) be any countable family of nonempty open sets in \( \Xi_p \). Enumerate \( \mathcal{B} \) as \( \langle B_k \rangle_{k \in \omega} \). For each \( k \in \omega \) choose \( c(k) \in B_k \) with \( c(k)_k \neq p_k \). Then

\[
U = \Xi_p \cap \Pi_{k \in \omega} (\mathbb{Q} - \{c(k)_k\})
\]

is a nonempty (for \( p \in U \)) open set in \( \Xi_p \) which does not include any
member of $\mathcal{B}$ (for $c(k) \not\in U$ for all $k \in \omega$). Hence $\mathcal{B}$ is not a $\pi$-base.

6. Completeness. By a theorem of Borges [B1] and, independently, Okuyama [O1], a space is metrizable iff it is a paracompact $p$-space with a $G_\delta$-diagonal, in particular, iff it is a stratifiable $p$-space. So every nonmetrizable stratifiable space is uncompletable in the sense that it cannot be embedded in a stratifiable Čech-complete space.

On the other hand there is a nonmetrizable stratifiable space $X$ which has a base $\mathcal{B}$, consisting of clopen sets, such that every centered subfamily of $\mathcal{B}$ has nonempty intersection (see [AL] for a survey of completeness properties of this sort): Let $X$ be the subspace $\mathbb{N} \cup \{p\}$ of $\beta\mathbb{N}$ for some $p \in \beta\mathbb{N}$. (The subspace $P \cup Q$ of [vD2, 2.3] would be a first countable such space.)

REFERENCES


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