BEHAVIOR OF MAXIMALLY DEFINED SOLUTIONS 
OF A NONLINEAR VOLterra EQUATION

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Abstract. This paper is concerned with the behavior of solutions of an n-dimensional nonlinear Volterra integral equation

\[ x(t) = f(t) + \int_0^t g(t, s, x(s))\,ds, \quad t > 0. \]

In particular, sufficient conditions for a solution \( x(t) \) on its maximal interval of existence \( [0, T) \) to possess the property that \( |x(t)| \to +\infty \) as \( t \to T^- \) are obtained. Thus these additional conditions give a positive answer to the problem posed by Miller [3, p. 145]. One can construct examples, satisfying the hypotheses given in [3], which provide a negative answer to this problem, see Artstein [1, Appendix A].

1. Introduction. In this paper we investigate the behavior of solutions of the \( n \)-dimensional nonlinear Volterra integral equation

\[ x(t) = f(t) + \int_0^t g(t, s, x(s))\,ds, \]

where \( t \) belongs to a half open interval \( [0, \eta) \), and the functions \( x, f \) and \( g \) have values in \( \mathbb{R}^n \), the real \( n \)-dimensional space. In particular, we shall investigate the behavior of \( \lim |x(t)| \) as \( t \to T^- \) where \( [0, T) \) is the maximal interval of existence for the solution \( x(t) \). The function \( f \) is assumed to be continuous on \( R^+ = [0, \infty) \), and we assume that the function \( g \) satisfies the following hypotheses:

(H1) \hspace{1em} The function \( g \) is defined for all \( (t, s, x) \in R^+ \times R^+ \times R^n \), \( g(t, s, x) = 0 \) whenever \( s > t \) and \( x \in R^n \); moreover, \( g(t, s, x) \) is measurable in \( s \) on \( [0, \eta] \) for each \( (t, x) \in R^+ \times R^n \), and \( g(t, s, x) \) is continuous in \( x \) for each fixed pair \( (t, s) \in R^+ \times R^+ \).

(H2) \hspace{1em} For each real number \( K > 0 \) and each bounded subset \( B \) of \( R^n \) there exists a measurable function \( m \) such that

\[ |g(t, s, x)| < m(t, s) \quad (0 < s < t < K, x \in B) \]

and

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\[ \sup \left\{ \int_0^t m(t, s) \, ds : 0 \leq t < K \right\} < \infty. \]

(H3) For each compact subinterval \( J \) of \( R^+ \), each bounded set \( B \) in \( R^n \) and each \( t_0 \) in \( R^+ \),

\[ \sup \left\{ \int_J \left| g(t, s, \varphi(s)) - g(t_0, s, \varphi(s)) \right| \, ds : \varphi \in C(J; B) \right\} \to 0 \]

as \( t \to t_0 \).

We have from [3] the following information about equation (E), when (H1) – (H3) are satisfied.

(C1) There exist a number \( \beta > 0 \) and a continuous function \( x(t) \) such that \( x(t) \) satisfies equation (E) on \([0, \beta]\).

(C2) If \( x(t) \) is a bounded continuous solution of (E) for \( t \in [0, \alpha) \), where \( \alpha < +\infty \), then \( x(t) \) can be extended as a continuous solution of (E) to an interval \([0, \alpha_0]\) where \( \alpha_0 > \alpha \).

(C3) Each continuous solution of (E) can be extended to the right to obtain a maximally defined solution (i.e. a continuous solution \( x(t) \) of (E) defined on an interval \([0, \alpha) \) where either \( \alpha = +\infty \) or \( \lim \sup |x(t)| = \infty \) as \( t \to \alpha^- \). The interval \([0, \alpha]\) is called the maximal interval of existence for the solution \( x(t) \).)

In [1], Artstein obtained these results under somewhat weaker but similar assumptions on the function \( g \) in equation (E). Herdman [2] obtained these results under different hypotheses.

From (C3) we see that if \( x(t) \) is a continuous solution of (E) on \([0, \eta]\) which cannot be extended, then either \( \eta = +\infty \) or \( \lim \sup |x(t)| = \infty \) as \( t \to \eta^- \). Artstein [1, Appendix A] constructed an example \( x: [0, 1) \to \mathbb{R} \) where \( \lim \sup |x(t)| = +\infty \) as \( t \to 1^- \) but \( \lim |x(t)| \neq \infty \) as \( t \to 1^- \). This example satisfied hypothesis (H1)-(H3) of this paper and (H7) of [3]. Thus, it is an answer to the open problem posed by Miller [3, Problem 15, p. 145]. This example gave a negative answer for the open problem.

The purpose of the present paper is to present sufficient conditions for a continuous solution \( x(t) \) of (E) on its maximal interval of existence \([0, T)\) to possess the property that \( |x(t)| \to \infty \) as \( t \to T^- \). In stating additional hypothesis for the function \( g \), we need the following vector notation. For a general \( n \)-vector \( M = (M_a) \) the symbol \( M \cdot > 0 \), \( (M \cdot > 0) \), signifies that the elements of \( M \) are real, and \( M_a > 0 \), \( (M_a > 0) \), for \( a = 1, \ldots, n \). The notation \( M \cdot > \cdot N \) is used to signify \( M - N \cdot > 0 \), with similar meaning for \( M \cdot > \cdot N \), \( M \cdot < \cdot N \) and \( M \cdot < \cdot N \). The symbol 0 is used indiscriminately for the zero vector of any dimension.

We now list two additional hypotheses for the function \( g \) in equation (E).
For each $\alpha > 0$ there exists a constant $n$-vector $K_\alpha \cdot > 0$ such that $g(s, s, x) \cdot > -K_\alpha$ for $(s, x) \in [0, \alpha] \times \mathbb{R}^n$.

For each $\alpha > 0$ there exists a constant $n$-vector $L_\alpha \cdot > 0$ such that

$$g(t, s, x) - g(\tau, s, x) \cdot > -L_\alpha[t - \tau]$$

for $0 < s < \tau < t < \alpha$ and $x \in \mathbb{R}^n$.

Our main result is the following theorem.

**Theorem 1.** Let the function $g$ of (E) satisfy (H4) and (H5). If $x(t)$ is a maximally defined solution of equation (E), with maximal interval of existence $[0, T)$ where $T < \infty$ then $\lim |x(t)| = +\infty$, as $t \to T^-$.

By way of comments, we note that the existence of a maximally defined solution is part of the hypothesis. Applying the results of [3], mentioned earlier in this paper, we see that equation (E) has a local solution and each solution can be extended to obtain a maximally defined solution. An example where the maximal interval of existence is of the form $[0, T), T < +\infty$ will be given in §4.

To conclude this section we state a lemma which will be used in the proof of Theorem 1.

**Lemma 1.** Let $y : [0, T) \to \mathbb{R}^n$, where $0 < T < +\infty$, be a continuous function satisfying

1. \(\lim_{t \to T^-}|y(t)| \neq +\infty,\)
2. there exists a constant $n$-vector $B \cdot > 0$ such that $y(t) - y(\tau) \cdot > -B(t - \tau)$ for $0 < \tau < t < T$.

Then the limit of $y(t)$ as $t \to T^-$ exists.

**2. Proof of Lemma 1.** Attention will be focused upon the case where $y$ is a real valued function. In particular, the inequality, $\cdot > \cdot$, in (1.2) will now be the usual, $>$, inequality for real values. The proof for the $n$-dimensional vector equation will follow immediately.

The hypothesis readily implies that there exists a sequence $\{t_n\}$, where $0 < t_n < t_{n+1} < T$ for $n = 1, 2, \ldots$, and a point $y_0$ such that $\{t_n\}$ converges to $T$ and $y(t_n)$ converges to $y_0$ as $n$ approaches $+\infty$.

Let $\varepsilon > 0$ be given, and let $\{\hat{t}_j\}$ be an arbitrary increasing sequence of values in $[0, T)$ such that $\hat{t}_j \to T^-$ as $j \to +\infty$. We choose $M_1$, sufficiently large, such that $|y(t_n) - y_0| < \varepsilon/3$ and $0 < T - t_n < \varepsilon/3(B + 1)$ whenever $n > M_1$. Now pick $M_2$ such that $t_{M_1} < \hat{t}_j < T$ for all $j > M_2$. It is to be noted that for each $j > M_2$ there is a $\eta_j > M_1$ such that $t_{M_1} < \hat{t}_j < t_{\eta_j}$, where $t_{\eta_j} \in \{t_n\}$.

In view of inequality (1.2), we have that

$$y(t_{\eta_j}) - y(\hat{t}_j) > -B(t_{\eta_j} - \hat{t}_j) > -B\varepsilon/3(B + 1) > -\varepsilon/3$$

for all $j > M_2$. Thus it follows that
for \( j > M_2 \). On the other hand, we have that
\[
y (t_{M_2}) > \frac{e}{3} (B + 1) > -\epsilon / 3
\]
for \( j > M_2 \). Consequently, it follows that
\[
(2.2) \quad y (t) \geq y (t_{M_2}) - \epsilon / 3 > y_0 - \epsilon
\]
for \( j > M_2 \). In particular, inequalities (2.1) and (2.2) imply that the sequence \( \{ y (t_j) \} \) converges to \( y_0 \) as \( j \to +\infty \). In view of the arbitrariness of the increasing sequence \( \{ t_i \} \) converging to \( T \), we have that the limit of \( y (t) \), as \( t \to T^- \), equals \( y_0 \in \mathbb{R} \).

3. **Proof of Theorem 1.** Define \( y (t) \) by
\[
(3.1) \quad y (t) = x (t) - f (t) \quad \text{for} \quad t \in [0, T).
\]
From (3.1) and the hypothesis it follows that \( y \in C ([0, T); \mathbb{R}^n) \).

To prove Theorem 1 we assume that the conclusion does not hold, that is,
\[
\lim |x (t)| = \gamma + 00 \quad \text{as} \quad t \to T^-.
\]
It is evident from (3.1) and the above assumption that \( \lim |y (t)| \neq +\infty \) as \( t \to T^- \).

We claim that
\[
(3.2) \quad \lim x (t) \quad \text{as} \quad t \to T^- \quad \text{exists}.
\]
As a first step in establishing (3.2) we assert that the function \( y \) defined by (3.1) satisfies the hypothesis of Lemma 1.

Let \( t \) and \( \tau \) be elements of \([0, T)\) satisfying \( 0 < \tau < t < T \). In view of (3.1) (H4) and (H5) one has
\[
y (t) - y (\tau) = x (t) - f (t) - x (\tau) + f (\tau)
\]
\[
= \int_0^t g (t, s, x (s)) ds - \int_0^\tau g (\tau, s, x (s)) ds
\]
\[
= \int_0^\tau \{ g (t, s, x (s)) - g (\tau, s, x (s)) \} ds
\]
\[
+ \int_\tau^t \{ g (t, s, x (s)) - g (s, s, x (s)) \} + \int_\tau^t g (s, s, x (s)) ds
\]
\[
\cdot > \cdot \int_0^\tau -L_T (t - s) ds + \int_\tau^t -L_T (t - s) ds + \int_\tau^t -K_T ds
\]
\[
\cdot > \cdot \int_\tau^t (t - \tau) - L_T \frac{(t - \tau)^2}{2} - K_T (t - \tau)
\]
\[
\cdot > \cdot (t - \tau) [L_T (T) - L_T (T) - K_T].
\]
Thus it follows that
\[
(3.3) \quad y (t) - y (\tau) \cdot > \cdot -B (t - \tau)
\]
for \( 0 < \tau < t < T \), where \( B = 2TL_T + K_T \cdot > \cdot 0 \). As a consequence of Lemma 1, it follows that the limit of \( |y (t)| \) as \( t \to T^- \) exists. Since \( f \) is a
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A continuous function on \([0, T)\) we have that limit of \(x(t)\) as \(t \to T^-\) exists. Thus (3.2) is established.

It is an elementary consequence of (3.2) that \(x(t)\) is bounded on \([0, T)\). In view of the result (C2) we have that there is a \(T_0 > T\) such that \(x(t)\) can be extended, continuously, as a solution of (E) on the interval \([0, T_0]\). This violates the interval \([0, T)\) being the maximal interval of existence for the solution \(x(t)\). From this contradiction we finally deduce that \(\|x(t)\| \to +\infty\) as \(t \to T^-\).

To conclude this section we note that the lower Lipschitz condition of (H5) and the lower bound of (H4) can be replaced by a corresponding upper Lipschitz condition and a corresponding upper bound. In particular, the conclusion of Theorem 1 holds when (H4) and (H5) are replaced by

For each \(\alpha > 0\) there exist constants \(k_{\alpha i} > 0\), \(l_{\alpha i} > 0\), \(i \in \mathbb{N} = \{1, 2, 3, \ldots, n\}\), and sets \(I_1, I_2\) such that

(i) \(I_1 \cup I_2 = \mathbb{N}, I_1 \cap I_2 = \emptyset\);
(ii) \(g_i(s, s, x) > -k_{\alpha i}\) for \((s, x) \in [0, \alpha] \times \mathbb{R}^n\) and \(i \in I_1\);
(iii) \(g_i(s, s, x) < k_{\alpha i}\) for \((s, x) \in [0, \alpha] \times \mathbb{R}^n\) and \(i \in I_2\);
(iv) \(g_i(t, s, x) - g_i(\tau, s, x) > -l_{\alpha i}[t - \tau]\) for \(0 < s < \tau < t < \alpha, x \in \mathbb{R}^n, i \in I_1\);
(v) \(g_i(t, s, x) - g_i(\tau, s, x) < l_{\alpha i}[t - \tau]\) for \(0 < \tau < t < \alpha, x \in \mathbb{R}^n, i \in I_2\).

4. Remarks. As an immediate consequence of Theorem 1 we have the following corollary. Again, we are assuming (H1), (H2) and (H3).

**Corollary 1.** Let

\[
\begin{align*}
(4.1) & \quad f' \in C([0, T], \mathbb{R}^n), \quad (\cdot = d/dt); \\
(4.2) & \quad g_i(t, s, x) \cdot > -M_T, \text{ where } M_T > 0, 0 < s < t < T, x \in \mathbb{R}^n; \\
(4.3) & \quad g(t, t, x) \cdot > -K, \text{ where } K > 0; \\
(4.4) & \quad \frac{d}{dt} \left( \int_0^t g(t, s, x(s)) ds \right) = g(t, t, x(t)) + \int_0^t g_i(t, s, x(s)) ds 
\end{align*}
\]

for \(t \in [0, T]\) and let \(x(t)\) be a maximally defined solution of (E) on \([0, T]\). Then \(\|x(t)\| \to +\infty\) as \(t \to T^-\).

To realize this Corollary one can take the derivative of \(x(t)\) to obtain

\[
x'(t) = f'(t) + g(t, t, x(t)) + \int_0^t g_i(t, s, x(s)) ds, \quad t \in [0, T].
\]

Applying conditions (4.1), (4.2) and (4.3) it follows that

\[
(4.5) \quad x'(t) \cdot > -B - K - M_T T \quad \text{for } t \in [0, T)
\]

where \(B\) is the constant \(n\) vector \((B_i), B_i = \sup \{|f'(t)|: t \in [0, T]\}\).

Inequality (4.5) together with reasoning similar to that given in the proof of Theorem 1 completes the proof.

Consider the equation
\( x(t) = x_0 + K \int_0^t (t - s)x(s)x^{1+a} \, ds \)

where \( x_0, K \) and \( a \) are positive constants. Equation (4.6) can be solved and the solution \( x(t) \) is defined implicitly by

\( \left( \frac{2K}{2 + a} \right)^{1/2} t = \frac{1}{\sqrt{x_0}} \int_1^{x(t)/x_0} (s^{2+a} - 1)^{-1/2} \, ds. \)

As seen in Miller [3, pp. 46-48], solutions of equation (4.6) have finite escape times. The maximal interval of existence for a solution defined by (4.7) is seen to be \([0, T)\) where \( T \) satisfies

\( \left( \frac{2K}{2 + a} \right)^{1/2} T = x_0^{-a/2} \int_1^\infty (s^{2+a} - 1)^{-1/2} \, ds. \)

We note that the kernel function

\[ g(t, s, x) = K(t - s)x^{1+a} \]

for equation (4.6) satisfies hypotheses (H4) and (H5) with \( K_T = 0 \) and \( L_T = 0 \). Therefore, Theorem 1 implies that \( |x(t)| \to \infty \) as \( t \to T^- \).

**References**


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