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BEHAVIOR OF MAXIMALLY DEFINED SOLUTIONS OF A NONLINEAR VOLTERRA EQUATION

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ABSTRACT. This paper is concerned with the behavior of solutions of an n-dimensional nonlinear Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s))ds, \qquad t \geq 0.$$

In particular, sufficient conditions for a solution $x(t)$ on its maximal interval of existence $[0, T)$ to possess the property that $|x(t)| \to +\infty$ as $t \to T^-$ are obtained. Thus these additional conditions give a positive answer to the problem posed by Miller [3, p. 145]. One can construct examples, satisfying the hypotheses given in [3], which provide a negative answer to this problem, see Artstein [1, Appendix A].

1. **Introduction.** In this paper we investigate the behavior of solutions of the n-dimensional nonlinear Volterra integral equation

$$\text{(E)} \qquad x(t) = f(t) + \int_0^t g(t, s, x(s))ds,$$

where t belongs to a half open interval $[0, \eta)$, and the functions x, f and g have values in R^n, the real n-dimensional space. In particular, we shall investigate the behavior of $\lim|x(t)|$ as $t \to T^-$ where $[0, T)$ is the maximal interval of existence for the solution $x(t)$. The function f is assumed to be continuous on $R^+ = [0, \infty)$, and we assume that the function g satisfies the following hypotheses:

(H1) The function g is defined for all $(t, s, x) \in R^+ \times R^+ \times R^n$, $g(t, s, x) = 0$ whenever $s > t$ and $x \in R^n$; moreover, $g(t, s, x)$ is measurable in s on $[0, t]$ for each $(t, x) \in R^+ \times R^n$, and $g(t, s, x)$ is continuous in x for each fixed pair $(t, s) \in R^+ \times R^+$.

(H2) For each real number $K > 0$ and each bounded subset B of R^n there exists a measurable function m such that

$$|g(t, s, x)| \leq m(t, s) \qquad (0 \leq s \leq t \leq K, x \in B)$$

and

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$$\sup\left\{\int_0^t m(t,s)ds\colon 0 \leqslant t \leqslant K\right\} < \infty.$$

(H3) For each compact subinterval J of R^+, each bounded set B in R^n and each t_0 in R^+,

$$\sup\left\{\int_J |g(t,s,\varphi(s)) - g(t_0,s,\varphi(s))|ds\colon \varphi \in C(J;B)\right\} \to 0$$

as $t \to t_0$.

We have from [3] the following information about equation (E), when (H1) – (H3) are satisfied.

(C1) There exist a number $\beta > 0$ and a continuous function $x(t)$ such that $x(t)$ satisfies equation (E) on $[0, \beta]$.

(C2) If $x(t)$ is a bounded continuous solution of (E) for $t \in [0, \alpha)$, where $\alpha < +\infty$, then $x(t)$ can be extended as a continuous solution of (E) to an interval $[0, \alpha_0]$ where $\alpha_0 > \alpha$.

(C3) Each continuous solution of (E) can be extended to the right to obtain a maximally defined solution (i.e. a continuous solution $x(t)$ of (E) defined on an interval $[0, \alpha)$ where either $\alpha = +\infty$ or $\limsup |x(t)| = \infty$ as $t \to \alpha^-$. The interval $[0, \alpha)$ is called the maximal interval of existence for the solution $x(t)$.)

In [1], Artstein obtained these results under somewhat weaker but similar assumptions on the function g in equation (E). Herdman [2] obtained these results under different hypotheses.

From (C3) we see that if $x(t)$ is a continuous solution of (E) on $[0, \eta)$ which cannot be extended, then either $\eta = +\infty$ or $\limsup |x(t)| = \infty$ as $t \to \eta^-$. Artstein [1, Appendix A] constructed an example $x\colon [0, 1) \to R$ where $\limsup|x(t)| = +\infty$ as $t \to 1^-$ but $\lim|x(t)| \neq \infty$ as $t \to 1^-$. This example satisfied hypothesis (H1)–(H3) of this paper and (H7) of [3]. Thus, it is an answer to the open problem posed by Miller [3, Problem 15, p. 145]. This example gave a negative answer for the open problem.

The purpose of the present paper is to present sufficient conditions for a continuous solution $x(t)$ of (E) on its maximal interval of existence $[0, T)$ to possess the property that $|x(t)| \to \infty$ as $t \to T^-$. In stating additional hypothesis for the function g, we need the following vector notation. For a general n-vector $M = (M_\alpha)$ the symbol $M \cdot \geqslant \cdot 0$, $\{M \cdot > \cdot 0\}$, signifies that the elements of M are real, and $M_\alpha \geqslant 0$, $\{M_\alpha > 0\}$, for $\alpha = 1, \ldots, n$. The notation $M \cdot \geqslant \cdot N$ is used to signify $M - N \cdot \geqslant \cdot 0$, with similar meaning for $M \cdot > \cdot N$, $M \cdot < \cdot N$ and $M \cdot \leqslant \cdot N$. The symbol 0 is used indiscriminately for the zero vector of any dimension.

We now list two additional hypotheses for the function g in equation (E).

(H4) For each $\alpha > 0$ there exists a constant n-vector $K_\alpha \cdot \geqslant \cdot 0$ such that $g(s, s, x) \cdot \geqslant \cdot -K_\alpha$ for $(s, x) \in [0, \alpha] \times R^n$.

(H5) For each $\alpha > 0$ there exists a constant n-vector $L_\alpha \cdot \geqslant \cdot 0$ such that

$$g(t, s, x) - g(\tau, s, x) \cdot \geqslant \cdot -L_\alpha[t - \tau]$$

for $0 \leqslant s \leqslant \tau \leqslant t \leqslant \alpha$ and $x \in R^n$.

Our main result is the following theorem.

THEOREM 1. *Let the function g of* (E) *satisfy* (H4) *and* (H5). *If $x(t)$ is a maximally defined solution of equation* (E), *with maximal interval of existence* $[0, T)$ *where $T < \infty$ then* $\lim|x(t)| = +\infty$, *as $t \to T^-$.*

By way of comments, we note that the existence of a maximally defined solution is part of the hypothesis. Applying the results of [3], mentioned earlier in this paper, we see that equation (E) has a local solution and each solution can be extended to obtain a maximally defined solution. An example where the maximal interval of existence is of the form $[0, T)$, $T < +\infty$ will be given in §4.

To conclude this section we state a lemma which will be used in the proof of Theorem 1.

LEMMA 1. *Let y: $[0, T) \to R^n$, where $0 < T < +\infty$, be a continuous function satisfying*

(1.1) $\lim_{t \to T^-}|y(t)| \neq +\infty$,

(1.2) *there exists a constant n-vector $B \cdot \geqslant \cdot 0$ such that $y(t) - y(\tau) \cdot \geqslant \cdot -B(t - \tau)$ for $0 \leqslant \tau \leqslant t < T$.*

Then the limit of $y(t)$ as $t \to T^-$ exists.

2. **Proof of Lemma** 1. Attention will be focused upon the case where y is a real valued function. In particular, the inequality, $\cdot \geqslant \cdot$, in (1.2) will now be the usual, $\geqslant$, inequality for real values. The proof for the n-dimensional vector equation will follow immediately.

The hypothesis readily implies that there exists a sequence $\{t_n\}$, where $0 < t_n < t_{n+1} < T$ for $n = 1, 2, \ldots,$ and a point y_0 such that $\{t_n\}$ converges to T and $y(t_n)$ converges to y_0 as n approaches $+\infty$.

Let $\varepsilon > 0$ be given, and let $\{\hat{t}_j\}$ be an arbitrary increasing sequence of values in $[0, T)$ such that $\hat{t}_j \to T^-$ as $j \to +\infty$. We choose M_1, sufficiently large, such that $|y(t_n) - y_0| < \varepsilon/3$ and $0 < T - t_n < \varepsilon/3(B + 1)$ whenever $n \geqslant M_1$. Now pick M_2 such that $t_{M_1} \leqslant \hat{t}_j < T$ for all $j \geqslant M_2$. It is to be noted that for each $j \geqslant M_2$ there is a $n_j > M_1$ such that $t_{M_1} \leqslant \hat{t}_j < t_{n_j}$, where $t_{n_j} \in \{t_n\}$.

In view of inequality (1.2), we have that

$$y(t_{n_j}) - y(\hat{t}_j) \geqslant -B(t_{n_j} - \hat{t}_j) > -B\varepsilon/3(B + 1) > -\varepsilon/3$$

for all $j \geqslant M_2$. Thus it follows that

$$y(\hat{t}_j) < y(t_{n_j}) + \varepsilon/3 < y_0 + \varepsilon \tag{2.1}$$

for $j \geqslant M_2$. On the other hand,we have that

$$y(\hat{t}_j) - y(t_{M_1}) \geqslant -B(\hat{t}_j - t_{M_1}) \geqslant -B\varepsilon/3(B+1) \geqslant -\varepsilon/3$$

for $j \geqslant M_2$. Consequently, it follows that

$$y(\hat{t}_j) \geqslant y(t_{M_1}) - \varepsilon/3 > y_0 - \varepsilon \tag{2.2}$$

for $j \geqslant M_2$. In particular, inequalities (2.1) and (2.2) imply that the sequence $\{y(\hat{t}_j)\}$ converges to y_0 as $j \to +\infty$. In view of the arbitrariness of the increasing sequence $\{\hat{t}_j\}$ converging to T, we have that the limit of $y(t)$, as $t \to T^-$, equals $y_0 \in R$.

3. **Proof of Theorem 1.** Define $y(t)$ by

$$y(t) = x(t) - f(t) \quad \text{for } t \in [0, T). \tag{3.1}$$

From (3.1) and the hypothesis it follows that $y \in C([0, T); R^n)$.

To prove Theorem 1 we assume that the conclusion does not hold, that is, $\lim|x(t)| \neq +\infty$ as $t \to T^-$. It is evident from (3.1) and the above assumption that $\lim|y(t)| \neq +\infty$ as $t \to T^-$.

We claim that

$$\lim x(t) \quad \text{as } t \to T^- \text{ exists.} \tag{3.2}$$

As a first step in establishing (3.2) we assert that the function y defined by (3.1) satisfies the hypothesis of Lemma 1.

Let t and τ be elements of $[0, T)$ satisfying $0 \leqslant \tau \leqslant t < T$. In view of (3.1) (H4) and (H5) one has

$$\begin{aligned}
y(t) - y(\tau) &= x(t) - f(t) - x(\tau) + f(\tau) \\
&= \int_0^t g(t, s, x(s))ds - \int_0^\tau g(\tau, s, x(s))ds \\
&= \int_0^\tau \{ g(t, s, x(s)) - g(\tau, s, x(s)) \} ds \\
&\quad + \int_\tau^t \{ g(t, s, x(s)) - g(s, s, x(s)) \} + \int_\tau^t g(s, s, x(s))ds \\
&\cdot\geqslant\cdot \int_0^\tau -L_T(t-\tau)ds + \int_\tau^t -L_T(t-s)ds + \int_\tau^t -K_T ds \\
&\cdot\geqslant\cdot -L_T(\tau)(t-\tau) - L_T\frac{(t-\tau)^2}{2} - K_T(t-\tau) \\
&\cdot\geqslant\cdot (t-\tau)[-L_T(T) - L_T(T) - K_T].
\end{aligned}$$

Thus it follows that

$$y(t) - y(\tau)\cdot \geqslant \cdot -B(t-\tau) \tag{3.3}$$

for $0 \leqslant \tau \leqslant t < T$, where $B = 2TL_T + K_T\cdot \geqslant \cdot 0$. As a consequence of Lemma 1, it follows that the limit of $|y(t)|$ as $t \to T^-$ exists. Since f is a

continuous function on $[0, T)$ we have that limit of $x(t)$ as $t \to T^-$ exists. Thus (3.2) is established.

It is an elementary consequence of (3.2) that $x(t)$ is bounded on $[0, T)$. In view of the result (C2) we have that there is a $T_0 > T$ such that $x(t)$ can be extended, continuously, as a solution of (E) on the interval $[0, T_0]$. This violates the interval $[0, T)$ being the maximal interval of existence for the solution $x(t)$. From this contradiction we finally deduce that $|x(t)| \to +\infty$ as $t \to T^-$.

To conclude this section we note that the lower Lipschitz condition of (H5) and the lower bound of (H4) can be replaced by a corresponding upper Lipschitz condition and a corresponding upper bound. In particular, the conclusion of Theorem 1 holds when (H4) and (H5) are replaced by

For each $\alpha > 0$ there exist constants $k_{\alpha i} \geqslant 0$, $l_{\alpha i} \geqslant 0$, $i \in N = \{1, 2, 3, \ldots, n\}$ and sets I_1, I_2 such that
(i) $I_1 \cup I_2 = N, I_1 \cap I_2 = \varnothing$;
(ii) $g_i(s, s, x) \geqslant -k_{\alpha i}$ for $(s, x) \in [0, \alpha] \times R^n$ and $i \in I_1$;
(iii) $g_i(s, s, x) \leqslant k_{\alpha i}$ for $(s, x) \in [0, \alpha] \times R^n$ and $i \in I_2$;
(iv) $g_i(t, s, x) - g_i(\tau, s, x) \geqslant -l_{\alpha i}[t - \tau]$ for $0 \leqslant s \leqslant \tau \leqslant t \leqslant \alpha, x \in R^n, i \in I_1$;
(v) $g_i(t, s, x) - g_i(\tau, s, x) \leqslant l_{\alpha i}[t - \tau]$ for $0 \leqslant s \leqslant \tau \leqslant t \leqslant \alpha, x \in R^n, i \in I_2$.

4. **Remarks.** As an immediate consequence of Theorem 1 we have the following corollary. Again, we are assuming (H1), (H2) and (H3).

COROLLARY 1. *Let*

$$f' \in C([0, T], R^n), \qquad (' = d/dt); \tag{4.1}$$

$$g_t(t, s, x) \cdot \geqslant \cdot -M_T, \quad \text{where } M_T \cdot > \cdot 0, 0 \leqslant s \leqslant t < T, x \in R^n; \tag{4.2}$$

$$g(t, t, x) \cdot \geqslant \cdot -K, \quad \text{where } K \cdot \geqslant \cdot 0; \tag{4.3}$$

$$\frac{d}{dt}\left(\int_0^t g(t, s, x(s))ds\right) = g(t, t, x(t)) + \int_0^t g_t(t, s, x(s))ds \tag{4.4}$$

for $t \in [0, T)$ and let $x(t)$ be a maximally defined solution of (E) on $[0, T)$. Then $|x(t)| \to +\infty$ as $t \to T^-$.

To realize this Corollary one can take the derivative of $x(t)$ to obtain

$$x'(t) = f'(t) + g(t, t, x(t)) + \int_0^t g_t(t, s, x(s))\, ds, \qquad t \in [0, T).$$

Applying conditions (4.1), (4.2) and (4.3) it follows that

$$x'(t) \cdot \geqslant \cdot -B - K - M_T T \quad \text{for } t \in [0, T) \tag{4.5}$$

where B is the constant n vector (B_i), $B_i = \sup\{|f'(t)|: t \in [0, T]\}$.

Inequality (4.5) together with reasoning similar to that given in the proof of Theorem 1 completes the proof.

Consider the equation

$$x(t) = x_0 + K\int_0^t (t - s)x(s)^{1+a}ds \tag{4.6}$$

where x_0, K and a are positive constants. Equation (4.6) can be solved and the solution $x(t)$ is defined implicitly by

$$\left(\frac{2K}{2+a}\right)^{1/2} t = \frac{1}{\sqrt{x_0^a}} \int_1^{x(t)/x_0} (s^{2+a} - 1)^{-1/2} ds. \tag{4.7}$$

As seen *n* Miller [3, pp. 46–48], solutions of equation (4.6) have finite escape times. The maximal interval of existence for a solution defined by (4.7) is seen to be $[0, T)$ where T satisfies

$$\left(\frac{2K}{2+a}\right)^{1/2} T = x_0^{-a/2} \int_1^{\infty} (s^{2+a} - 1)^{-1/2} ds. \tag{4.8}$$

We note that the kernel function

$$g(t, s, x) = K(t - s)x^{1+a}$$

for equation (4.6) satisfies hypotheses (H4) and (H5) with $K_T = 0$ and $L_T = 0$. Therefore, Theorem 1 implies that $|x(t)| \to \infty$ as $t \to T^-$.

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