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DELTAS OF HOCHSCHILD DIMENSION ONE

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ABSTRACT. A theorem of Mitchell says that free categories have Hochschild dimension ≤ 1. The converse is shown to be true for all deltas (i.e. small skeletal categories whose only endomorphisms are the identities).

Throughout K will denote a nonzero commutative ring with identity, and $\mathfrak{M}$ will be the category of K-modules. Let $\mathbf{C}$ be a small category, and let $K\mathbf{C}(p, q)$ be the free K-module on the set $\mathbf{C}(p, q)$ of morphisms from p to q. Then $K\mathbf{C}$ can be regarded as an object of the functor category $\mathfrak{M}^{\mathbf{C}^{\mathrm{op}}\times\mathbf{C}}$, and its projective dimension is called the *K-Hochschild dimension* of $\mathbf{C}$ ($\dim_K \mathbf{C}$). Clearly $\dim_K \mathbf{C} = \dim_K \mathbf{C}^{\mathrm{op}}$. In [2, p. 62] it is shown that if $D \in \mathfrak{M}^{\mathbf{C}}$, then

$$\text{(1)} \qquad \operatorname{pd} D \leq \dim_K \mathbf{C} + \sup_{p\in\mathbf{C}} \operatorname{pd} D(p),$$

where pd denotes projective dimension. In particular, if D has free K-modules as values, then $\operatorname{pd} D \leq \dim_K \mathbf{C}$. When $\mathbf{C}$ is a group, $\dim_K \mathbf{C}$ is the same as the K-cohomological dimension of the group [1, p. 195]. Therefore by Stallings [3] and Swan [4] we know that $\dim_{\mathbf{Z}} \mathbf{C} \leq 1$ if and only if $\mathbf{C}$ is free as a group. In this paper we show that if $\mathbf{C}$ is a delta (i.e. a small skeletal category whose only endomorphisms are the identities) then $\dim_K \mathbf{C} \leq 1$ if and only if $\mathbf{C}$ is the free category generated by a directed graph. This completes a theorem of Mitchell [2, p. 151], who established it in case $\mathbf{C}$ is either a weak delta or a partially ordered set.

Henceforth $\mathbf{C}$ will denote a delta. If α is a morphism in $\mathbf{C}$ then we denote its domain and codomain by dom α and cod α, respectively. The *length* of α is defined to be $\sup\{k \mid \alpha = \alpha_1\alpha_2 \cdots \alpha_k,\ \alpha_i \neq 1\}$. If $D \in \mathfrak{M}^{\mathbf{C}}$ then we denote the image of $D(\alpha)$ by $\alpha D(p)$. If p and q are objects of $\mathbf{C}$, define $p \leq q$ if $\mathbf{C}(p, q) \neq \varnothing$. Because $\mathbf{C}$ is skeletal and all endomorphisms are identities, it follows that this is a partial order. For each $p \in \mathbf{C}$ let $S_p\colon \mathfrak{M} \to \mathfrak{M}^{\mathbf{C}}$ denote the left adjoint of the pth evaluation functor $D \mapsto D(p)$. Then S_p is given explicitly by

$$S_p(A)(q) = \bigoplus_{\mathbf{C}(p,q)} A.$$

Note that S_p preserves projectives since it has an exact right adjoint.

LEMMA 1. *Let $D \in \mathfrak{M}^{\mathbf{C}}$ be projective. If $\alpha_i\colon p_i \to q$, $i = 1, 2$, are morphisms*

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in **C**, *then*

$$\alpha_1 D(p_1) \cap \alpha_2 D(p_2) = \sum \alpha_1 \beta_1 D(\operatorname{dom} \beta_1), \tag{2}$$

where the sum is indexed by all β_1 *with codomain* p_1 *such that there exists* β_2 *making the following diagram commutative.*

$$\begin{tikzcd} & \bullet \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\ p_1 \arrow[dr, "\alpha_1"'] & & p_2 \arrow[dl, "\alpha_2"] \\ & q & \end{tikzcd}$$

PROOF. The right side of (2) is always contained in the left side, and the inclusion is a natural transformation of functors of the variable D. It follows that coproducts and retracts of objects satisfying (2) also satisfy (2). Clearly $S_p(A)$ satisfies (2) for all $A \in \mathfrak{M}$ and $p \in \mathbf{C}$. Consider the epimorphism

$$\pi\colon \bigoplus_{p \in \mathbf{C}} S_p(D(p)) \to D$$

where the pth coordinate of π is induced by the identity map. Since D is projective, π splits. Hence D, as a retract of a coproduct whose factors satisfy (2), also satisfy (2).

LEMMA 2. *Suppose* $\dim_K \mathbf{C} \leqslant 1$ *and the diagram below is commutative. If either* α_1 *and* α_2 *have length one or* $\operatorname{dom} \alpha_1 = \operatorname{dom} \alpha_2$, *then* $\alpha_1 = \alpha_2$.

$$\begin{tikzcd} & e \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & \\ p_1 \arrow[dr, "\alpha_1"'] & & p_2 \arrow[dl, "\alpha_2"] \\ & \bullet & \end{tikzcd}$$

PROOF. Let $E = S_e(K)$. Define $D \in \mathfrak{M}^{\mathbf{C}}$ as follows:

$$\begin{aligned} D(p) &= E(p) && \text{if there exists } \beta_i\colon p \to p_i,\ i = 1, 2, \\ & && \text{with } \alpha_1 \beta_1 = \alpha_2 \beta_2, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Consider the short exact sequence $0 \to N \to E \to^{\pi} D \to 0$ where π_p is the identity map whenever possible. Since $\dim_K \mathbf{C} \leqslant 1$, we have $\operatorname{pd} D \leqslant 1$. Therefore since E is projective, N is projective. By Lemma 1, we have

$$\alpha_1 N(p_1) \cap \alpha_2 N(p_2) = \sum_{\beta_1} \alpha_1 \beta_1 N(\operatorname{dom} \beta_1) = 0. \tag{3}$$

If $\alpha_1 \neq \alpha_2$ and if either α_1 and α_2 have length one or dom $\alpha_1 =$ dom α_2, then there exists no commutative square of the form

$$\begin{tikzcd}
& p \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\
p_1 \arrow[dr, "\alpha_1"'] & & p_2 \arrow[dl, "\alpha_2"] \\
& \bullet &
\end{tikzcd}$$

where p is either p_1 or p_2. Here we used the fact that all endomorphisms of $\mathbf{C}$ are identities. Hence $N(p_i) = E(p_i)$, $i = 1, 2$. Since E is projective,

$$\begin{aligned} \alpha_1 N(p_1) \cap \alpha_2 N(p_2) &= \alpha_1 E(p_1) \cap \alpha_2 E(P_2) \\ &= \sum_{\beta_1} \alpha_1 \beta_1 E(\text{dom}\, \beta_1) \supseteq \alpha_1 \gamma_1 E(e) \neq 0. \end{aligned}$$

This contradicts (3).

We need the following result [2, Corollary 36.11].

Lemma 3. *If* $\dim_K \mathbf{C} \leqslant 1$, *then every morphism of* $\mathbf{C}$ *is a composite of morphisms of length one.*

Theorem 4. $\dim_K \mathbf{C} \leqslant 1$ *if and only if* $\mathbf{C}$ *is free.*

Proof. The "if" part is known for any small category $\mathbf{C}$ [2, Corollary 28.3].

Suppose $\dim_K \mathbf{C} \leqslant 1$ and suppose there is a morphism that has two representations as composites of morphisms of length one, say, $x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_n$, $m \leqslant n$. We will show that $m = n$ and $x_i = y_i$, $i = 1, 2, \ldots, m$. This is clearly true if $m = 1$. Using the first part of Lemma 2 we see that $x_1 = y_1$. Then, by the dual of the second part of Lemma 2, $x_2 \cdots x_m = y_2 \cdots y_n$. Hence, by induction, $m = n$ and $x_i = y_i$, $2 \leqslant i \leqslant m$.

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