

## DELTAS OF HOCHSCHILD DIMENSION ONE

CHARLES CHING-AN CHENG

**ABSTRACT.** A theorem of Mitchell says that free categories have Hochschild dimension  $< 1$ . The converse is shown to be true for all deltas (i.e. small skeletal categories whose only endomorphisms are the identities).

Throughout  $K$  will denote a nonzero commutative ring with identity, and  $\mathcal{M}$  will be the category of  $K$ -modules. Let  $\mathbf{C}$  be a small category, and let  $K\mathbf{C}(p, q)$  be the free  $K$ -module on the set  $\mathbf{C}(p, q)$  of morphisms from  $p$  to  $q$ . Then  $K\mathbf{C}$  can be regarded as an object of the functor category  $\mathcal{M}^{\mathbf{C}^{\text{op}} \times \mathbf{C}}$ , and its projective dimension is called the  *$K$ -Hochschild dimension* of  $\mathbf{C}$  ( $\dim_K \mathbf{C}$ ). Clearly  $\dim_K \mathbf{C} = \dim_K \mathbf{C}^{\text{op}}$ . In [2, p. 62] it is shown that if  $D \in \mathcal{M}^{\mathbf{C}}$ , then

$$(1) \quad \text{pd } D \leq \dim_K \mathbf{C} + \sup_{p \in \mathbf{C}} \text{pd } D(p),$$

where  $\text{pd}$  denotes projective dimension. In particular, if  $D$  has free  $K$ -modules as values, then  $\text{pd } D \leq \dim_K \mathbf{C}$ . When  $\mathbf{C}$  is a group,  $\dim_K \mathbf{C}$  is the same as the  $K$ -cohomological dimension of the group [1, p. 195]. Therefore by Stallings [3] and Swan [4] we know that  $\dim_{\mathbb{Z}} \mathbf{C} \leq 1$  if and only if  $\mathbf{C}$  is free as a group. In this paper we show that if  $\mathbf{C}$  is a delta (i.e. a small skeletal category whose only endomorphisms are the identities) then  $\dim_K \mathbf{C} \leq 1$  if and only if  $\mathbf{C}$  is the free category generated by a directed graph. This completes a theorem of Mitchell [2, p. 151], who established it in case  $\mathbf{C}$  is either a weak delta or a partially ordered set.

Henceforth  $\mathbf{C}$  will denote a delta. If  $\alpha$  is a morphism in  $\mathbf{C}$  then we denote its domain and codomain by  $\text{dom } \alpha$  and  $\text{cod } \alpha$ , respectively. The *length* of  $\alpha$  is defined to be  $\sup\{k | \alpha = \alpha_1 \alpha_2 \cdots \alpha_k, \alpha_i \neq 1\}$ . If  $D \in \mathcal{M}^{\mathbf{C}}$  then we denote the image of  $D(\alpha)$  by  $\alpha D(p)$ . If  $p$  and  $q$  are objects of  $\mathbf{C}$ , define  $p \leq q$  if  $\mathbf{C}(p, q) \neq \emptyset$ . Because  $\mathbf{C}$  is skeletal and all endomorphisms are identities, it follows that this is a partial order. For each  $p \in \mathbf{C}$  let  $S_p: \mathcal{M} \rightarrow \mathcal{M}^{\mathbf{C}}$  denote the left adjoint of the  $p$ th evaluation functor  $D \mapsto D(p)$ . Then  $S_p$  is given explicitly by

$$S_p(A)(q) = \bigoplus_{\mathbf{C}(p, q)} A.$$

Note that  $S_p$  preserves projectives since it has an exact right adjoint.

**LEMMA 1.** *Let  $D \in \mathcal{M}^{\mathbf{C}}$  be projective. If  $\alpha_i: p_i \rightarrow q, i = 1, 2$ , are morphisms*

*Received by the editors November 19, 1976.*

*AMS (MOS) subject classifications (1970). Primary 18G20; Secondary 18B99.*

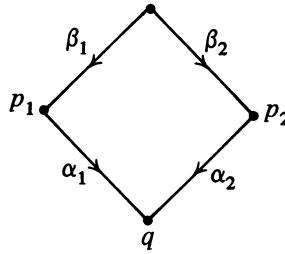
*Key words and phrases.* Delta, Hochschild dimension.

© American Mathematical Society 1978

in  $\mathbf{C}$ , then

$$(2) \quad \alpha_1 D(p_1) \cap \alpha_2 D(p_2) = \sum_{\beta_1} \alpha_1 \beta_1 D(\text{dom } \beta_1),$$

where the sum is indexed by all  $\beta_1$  with codomain  $p_1$  such that there exists  $\beta_2$  making the following diagram commutative.

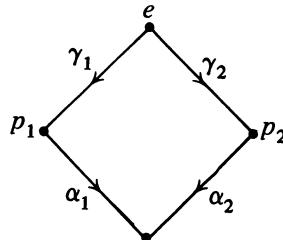


**PROOF.** The right side of (2) is always contained in the left side, and the inclusion is a natural transformation of functors of the variable  $D$ . It follows that coproducts and retracts of objects satisfying (2) also satisfy (2). Clearly  $S_p(A)$  satisfies (2) for all  $A \in \mathfrak{M}$  and  $p \in \mathbf{C}$ . Consider the epimorphism

$$\pi: \bigoplus_{p \in \mathbf{C}} S_p(D(p)) \rightarrow D$$

where the  $p$ th coordinate of  $\pi$  is induced by the identity map. Since  $D$  is projective,  $\pi$  splits. Hence  $D$ , as a retract of a coproduct whose factors satisfy (2), also satisfy (2).

**LEMMA 2.** Suppose  $\dim_K \mathbf{C} \leq 1$  and the diagram below is commutative. If either  $\alpha_1$  and  $\alpha_2$  have length one or  $\text{dom } \alpha_1 = \text{dom } \alpha_2$ , then  $\alpha_1 = \alpha_2$ .



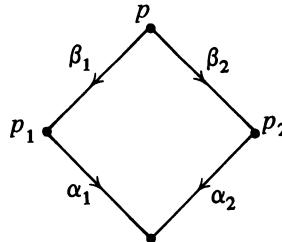
**PROOF.** Let  $E = S_e(K)$ . Define  $D \in \mathfrak{M}^{\mathbf{C}}$  as follows:

$$\begin{aligned} D(p) &= E(p) && \text{if there exists } \beta_i: p \rightarrow p_i, i = 1, 2, \\ &&& \text{with } \alpha_1 \beta_1 = \alpha_2 \beta_2, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Consider the short exact sequence  $0 \rightarrow N \rightarrow E \rightarrow^{\pi} D \rightarrow 0$  where  $\pi_p$  is the identity map whenever possible. Since  $\dim_K \mathbf{C} \leq 1$ , we have  $\text{pd } D \leq 1$ . Therefore since  $E$  is projective,  $N$  is projective. By Lemma 1, we have

$$(3) \quad \alpha_1 N(p_1) \cap \alpha_2 N(p_2) = \sum_{\beta_1} \alpha_1 \beta_1 N(\text{dom } \beta_1) = 0.$$

If  $\alpha_1 \neq \alpha_2$  and if either  $\alpha_1$  and  $\alpha_2$  have length one or  $\text{dom } \alpha_1 = \text{dom } \alpha_2$ , then there exists no commutative square of the form



where  $p$  is either  $p_1$  or  $p_2$ . Here we used the fact that all endomorphisms of  $\mathbf{C}$  are identities. Hence  $N(p_i) = E(p_i)$ ,  $i = 1, 2$ . Since  $E$  is projective,

$$\begin{aligned} \alpha_1 N(p_1) \cap \alpha_2 N(p_2) &= \alpha_1 E(p_1) \cap \alpha_2 E(p_2) \\ &= \sum_{\beta_1} \alpha_1 \beta_1 E(\text{dom } \beta_1) \supseteq \alpha_1 \gamma_1 E(e) \neq 0. \end{aligned}$$

This contradicts (3).

We need the following result [2, Corollary 36.11].

**LEMMA 3.** *If  $\dim_K \mathbf{C} \leq 1$ , then every morphism of  $\mathbf{C}$  is a composite of morphisms of length one.*

**THEOREM 4.**  $\dim_K \mathbf{C} \leq 1$  if and only if  $\mathbf{C}$  is free.

**PROOF.** The “if” part is known for any small category  $\mathbf{C}$  [2, Corollary 28.3].

Suppose  $\dim_K \mathbf{C} \leq 1$  and suppose there is a morphism that has two representations as composites of morphisms of length one, say,  $x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_n$ ,  $m < n$ . We will show that  $m = n$  and  $x_i = y_i$ ,  $i = 1, 2, \dots, m$ . This is clearly true if  $m = 1$ . Using the first part of Lemma 2 we see that  $x_1 = y_1$ . Then, by the dual of the second part of Lemma 2,  $x_2 \cdots x_m = y_2 \cdots y_n$ . Hence, by induction,  $m = n$  and  $x_i = y_i$ ,  $2 \leq i \leq m$ .

**ACKNOWLEDGEMENT.** I would like to thank the referee for helping to simplify the exposition of the paper.

#### REFERENCES

1. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.
2. B. Mitchell, *Rings with several objects*, Advances in Math. **8** (1972), 1–161.
3. J. R. Stallings, *On torsion free groups with infinitely many ends*, Ann. of Math. (2) **88** (1968), 312–334.
4. R. G. Swan, *Groups of cohomological dimension one*, J. Algebra **12** (1969), 585–610.

DEPARTMENT OF MATHEMATICS, OAKLAND UNIVERSITY, ROCHESTER, MICHIGAN 48063