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A CONTINUATION RESULT FOR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this note we show how the Conti-Wintner and Yoshizawa continuation results can be strengthened and combined to produce a flexible continuation theorem. That result is then applied to second and third order equations.

1. Introduction. We consider a system of ordinary differential equations

$$x' = f(t, x) \tag{1}$$

with $f: [0, \infty) \times R^n \to R^n$ and continuous. Then for each (t_0, x_0) in $[0, \infty) \times R^n$ there is at least one solution $x(t, t_0, x_0)$ defined on a right-maximal interval $[t_0, T)$, and, if $T < \infty$, then $\lim_{t\to T^-}|x(t)| = +\infty$.

There are two general results yielding $T = \infty$ which have had wide application. The first is known as the Wintner-Conti theorem [1], [4], [5], and the second may be traced back at least to Yoshizawa, but is given in a better form by Lakshmikantham and Leela [3, p. 135].

THEOREM 1. *Let* λ *and* ω *be continuous functions with* $\lambda: [0, \infty) \to [0, \infty)$, $\omega: [0, \infty) \to (0, \infty)$, *and* $\int_0^\infty [ds/\omega(s)] = \infty$. *If* $|f(t, x)| \leq \lambda(t)\omega(|x|)$ *on* $[0, \infty) \times R^n$, *then each solution of* (1) *can be continued for all future time.*

The virtue of this result over the next is that the functions λ and ω are often apparent from f itself, while the next result requires construction of a Liapunov function V. However, investigators have been very successful in constructing particular Liapunov functions, but these functions frequently fail to fulfill some condition of the result now given.

THEOREM 2. *Let* V *and* ϕ *be continuous functions,* $V: [0, \infty) \times R^n \to [0, \infty)$, $V(t, x) \to \infty$ *as* $|x| \to \infty$ *uniformly for* t *in compact sets,* V *locally Lipschitz in* x, $\phi: [0, \infty) \times [0, \infty) \to (-\infty, \infty)$, *and* $V'_{(1)}(t, x) \leq \phi(t, V)$ *on* $[0, \infty) \times R^n$. *If for each* $u_0 > 0$ *the maximal solution of*

$$u' = \phi(t, u), \qquad u(t_0) = u_0,$$

can be continued for all $t \geq t_0$, *then each solution of* (1) *exists for all future time.*

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In examples it frequently happens that $V(t, x) \nrightarrow \infty$ as $|x| \to \infty$, although V is unbounded along certain paths in R^n. Also, if $f = (f_1, \ldots, f_n)$, then for some i an inequality $|f_i(t, x)| \leq \lambda(t)\omega(|x|)$ holds for λ and ω satisfying Theorem 1, but for at least one i, the inequality fails.

With this in mind we combine the results and obtain a more flexible theorem.

2. An extension. Let V, g, r, h, η and ω be continuous functions with the following properties:

(a) V: $[0, \infty) \times R^n \to [0, \infty)$, V locally Lipschitz in x.

(b) g, h, and η: $[0, \infty) \to [0, \infty)$.

(c) r: $[0, \infty) \times [0, \infty) \to (-\infty, \infty)$ and for each $u_0 > 0$ the maximal solution of $[u' = r(t, u), u(t_0) = u_0]$ exists for all $t \geq t_0$.

(d) $V'_{(1)}(t, x) \leq -g(t, x) + r(t, V)$ on $[0, \infty) \times R^n$.

(e) ω: $[0, \infty) \to (0, \infty)$, $\int_0^\infty [ds/\omega(s)] = \infty$, and ω is monotone nondecreasing.

THEOREM 3. *Let* (a)–(e) *hold and suppose that for each* $T > 0$ *and each* i, *if*

(A) $|f_i(t, x)| \leq \eta(t)[g(t, x) + h(V(t, x)) + \omega(|x|)]$ *for* $0 \leq t \leq T$ *and* $x \in R^n$ *fails, then for each* $K > 0$ *there exists* $M > 0$ *such that either*

(B) $V(t, x) \leq K$, $0 \leq t \leq T$, *and* $x \in R^n$ *implies* $|f_i(t, x)| \leq M$, *or*

(C) $V(t, x) \leq K$ *and* $0 \leq t \leq T$ *implies* $|x_i| \leq M$ *holds*.

Then each solution of (1) *exists for all future time.*

PROOF. If the theorem is false then there is a solution $x(t)$ of (1) defined on $[t_0, T)$ with $T < \infty$ and $\lim_{t\to T^-}|x(t)| = +\infty$.

Let $u_0 = V(t_0, x(t_0))$ in (c) and conclude that $V(t, x(t)) \leq u(t, t_0, u_0)$, the maximal solution through (t_0, u_0), for $t_0 \leq t < T$. Thus, if $K = \max_{t_0 \leq t \leq T}|u(t, t_0, u_0)|$ then $V(t, x(t)) \leq K$ on $[t_0, T)$ and $h(V(t, x(t))) \leq H$ on $[t_0, T)$ for some $H > 0$.

Also, r is continuous and so there exists $Q > 0$ with $r(t, V(t, x(t))) < Q$ on $[t_0, T)$. We then have $V'(t, x(t)) \leq -g(t, x(t)) + Q$ so that

$$
\begin{aligned}
0 &\leq V(t, x(t)) \\
&\leq V(t_0, x(t_0)) + Q(T - t_0) - \int_{t_0}^{t} g(s, x(s))\, ds,
\end{aligned}
$$

yielding $\int_{t_0}^{T^-} g(s, x(s))\, ds < P$ for some $P > 0$.

Next, note that if (C) holds for some i, then $|x_i(t)| \leq M$ on $[t_0, T)$. If (B) holds for some i, then

$$
\begin{aligned}
|x_i(t)| &\leq |x_i(t_0)| + \int_{t_0}^{t} |f_i(s, x(s))|\, ds \\
&\leq |x_i(t_0)| + M(T - t_0).
\end{aligned}
$$

While if (B) and (C) fail, then (A) yields

$$
|x_i(t)| \leq |x_i(t_0)| + N\left[P + H(T - t_0) + \int_{t_0}^{t} \omega(|x(s)|)\, ds\right],
$$

where $N = \max_{t_0 < t < T} \eta(t)$. Thus, in any of these cases we have

$$|x_i(t)| < |x_i(t_0)| + S + \int_{t_0}^{t} N\omega(|x(s)|)\, ds$$

on $[t_0, T)$ for some $S > 0$.

We conclude that

$$|x(t)| < J + L\int_{t_0}^{t} \omega(|x(s)|)\, ds$$

on $[t_0, T)$ for some positive constants J and L. As ω is monotone nondecreasing, $|x(t)|$ is bounded on $[t_0, T)$ by the maximal solution of

$$v' = L\omega(v), \qquad v(t_0) = J$$

(cf. Hartman [2, p. 29]). However, that solution can be continued for all future time and so $|x(t)|$ is bounded on $[t_0, T)$. This completes the proof.

REMARK 1. Efforts to prove Theorem 3 without the monotonicity assumption on ω have failed, but it seems reasonable that the condition is not needed.

EXAMPLE 1. Consider the system of three equations,

$$x' = y, \quad y' = z, \quad z' = -\phi(x, y)z - \psi(y) - \gamma(x),$$

in which we have taken $(x, y, z) = (x_1, x_2, x_3)$ to avoid subscripts. It is assumed that all functions are continuous, $\phi(x, y) \geq 2a$ for some $a > 0$, $\gamma'(x)$ is bounded, $y\psi(y) > 0$ if $y \neq 0$, $x\gamma(x) > 0$ if $x \neq 0$, and $y\partial\phi(x, y)/\partial x \leq 0$.

Let $\Gamma(x) = \int_0^x \gamma(s)\, ds$, $G(y) = \int_0^y \psi(s)\, ds$, and assume $a\Gamma(x) + y\gamma(x) + G(y) \geq -S$ for some $S > 0$. Define

$$V(t, x, y, z) = (z + ay)^2/2 + a\Gamma(x) + y\gamma(x) + G(y) + a\int_0^y \{\phi(x, s) - a\}s\, ds + S$$

and obtain

$$\begin{aligned} V' &= -\{ay\psi(y) - y^2\gamma'(x)\} - \{\phi(x, y) - a\}z^2 \\ &\quad + ay\int_0^y s[\partial\phi(x, s)/\partial x]\, ds \\ &\leq dy^2 - az^2, \end{aligned}$$

as $\gamma'(x)$ is bounded and $\phi(x, y) \geq 2a$. But $\int_0^y \{\phi(x, s) - a\}s\, ds \geq ay^2/2$ and so $V' \leq -az^2 + \alpha V$ for some constant $\alpha > 0$.

Now V bounded yields y bounded and, hence, z is bounded. Thus, refer to (B) and (C) of Theorem 3 and note that $V(t, x) \leq K$ yields f_1 bounded, f_2 bounded, and x_3 bounded on an interval $t_0 \leq t < T$. Solutions are continuable.

We have taken $g(t, x, y, z) = az^2$, $h(V(t, x)) = 0$, and $r(t, s) = \alpha s$. As (A) was not used, it suffices to set $\eta(t) = \omega(s) = 1$.

EXAMPLE 2. Again, let $(x, y) = (x_1, x_2)$ and consider a system

$$x' = y - \phi(x), \qquad y' = -\psi(x) + e(t)$$

with all functions continuous, $\int_0^x \psi(s)\,ds \geq -P$, and $\psi(x)\phi(x) \geq -Q$ for some positive numbers P and Q. We assume that there is a nondecreasing continuous function λ: $[0, \infty) \to (0, \infty)$ with $|\phi(x)| \leq \psi(x)\phi(x) + P + \lambda(|x|)$, where $\int_0^\infty [ds/\{s + \lambda(s)\}] = \infty$.

Take $V = [y^2/2] + \int_0^x \psi(s)\,ds + P$ and obtain

$$\begin{aligned} V' &= -\psi(x)\phi(x) + ye(t) \\ &\leq -[\psi(x)\phi(x) + Q] + 2|e(t)|V + Q + |e(t)|. \end{aligned}$$

Then $r(t, s) = 2|e(t)|s + |e(t)| + Q$, $g(t, x) = \psi(x)\phi(x) + Q$, $\eta(t) = 1$, and $h(s) = 0$.

When $V \leq K$, then $y = x_2$ is bounded. We have

$$\begin{aligned} |f_1(t, x)| &= |y - \phi(x)| \leq |y| + |\phi(x)| \\ &\leq |y| + \psi(x)\phi(x) + P + \lambda(|x|) \\ &\leq \psi(x)\cdot\phi(x) + P + \omega(|x| + |y|), \end{aligned}$$

where $\omega(s) = s + \lambda(s)$.

As an instance of Example 2, consider the forced van der Pol equation

$$\begin{aligned} x' &= y - \varepsilon([x^3/3] - x) \qquad (\varepsilon > 0), \\ y' &= -x + e(t). \end{aligned}$$

We have $\phi(x)\psi(x) = \varepsilon x^2([x^2/3] - 1)$ so that $\lambda(|x|) = \varepsilon$ will suffice.

REMARK 2. The reader may note that Theorem 1 is often stated with $|f(t, x)| \leq \omega(t, |x|)$, where solutions of $r' = \omega(t, r)$ are continuable. This change can also be made in Theorem 3. Also, in applications it is sometimes conceptually easier to require Condition (A) of Theorem 3 only for those x for which $V(t, x) \leq K$; that change is also valid.

REMARK 3. Theorem 3 is fully applicable to delay differential equations

$$x' = f(t, x) + g(t, x(t - \tau(t)))$$

with $\tau(t) > 0$. For if one supposes a solution is defined on an interval $[t_0, T)$ with $\overline{\lim}_{t\to T^-}|x(t)| = +\infty$, then $x(t - \tau(t))$ is a bounded function on $[t_0, T]$ so that one is considering the continuation problem for the ordinary differential equation

$$x' = f(t, x) + e(t)$$

for $t_0 \leq t < T$ with $e(t) = g(t, x(t - \tau(t)))$. Our Example 2 is an instance of this.

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