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A CHARACTERISTIC PROPERTY OF THE SPHERE

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ABSTRACT. On an ovaloid S with Gaussian curvature $K > 0$ in Euclidean three-space E^3, the second fundamental form defines a nondegenerate Riemannian metric with curvature K_{II}. It is shown that S is a sphere if $K_{\mathrm{II}} = cH^sK^r$, where c, s and r are constants, H is the mean curvature of S and $0 < s < 1$.

A closed surface S in a Euclidean three-space E^3 with positive Gaussian curvature K possesses a positive definite second fundamental form II, if appropriately oriented. We denote by K_{II} the Gaussian curvature of the second fundamental form and by H the mean curvature of S. Many authors have been concerned with the problem of characterization of the sphere by the curvature of the second fundamental form. In [4] R. Schneider has shown that the constancy of the curvature K_{II} implies that S is a sphere. D. Koutroufiotis [3] has shown that S is a sphere if $K_{\mathrm{II}} = cK$ for some constant c or if $K_{\mathrm{II}} = \sqrt{K}$. In [5] U. Simon has given a new proof for a result in [4] and some other results. In [6] G. Stamou has shown that S is a sphere if $K_{\mathrm{II}} = cH\sqrt{K}$ or $K_{\mathrm{II}} = cH/\sqrt{K}$ for some constant c. Moreover, in [6] it was proved that S is a sphere if $K_{\mathrm{II}} = cK^r$ for some constants c and r. This result gives the results of [3] and [4] for appropriate c and r. In [2] Th. Koufogiorgos has shown that S is a sphere if $K_{\mathrm{II}} = cHK^r$ for some constants c and r. This result generalizes some results of [6]. Finally, Th. Hasanis [1] has shown that S is a sphere if $K_{\mathrm{II}} = c\sqrt{H}\,K^r$ for some constants c and r. The purpose of this paper is to prove a result which contains the above results as special cases. At first we state some preliminaries.

Let Γ^k_{ij}, ∇, respectively ${}_{\mathrm{II}}\Gamma^k_{ij}$, ∇_{II}, denote Christoffel symbols and the first Beltrami operator with respect to the first fundamental form I respective to the second fundamental form II. Let

$$T^k_{ij} = \Gamma^k_{ij} - {}_{\mathrm{II}}\Gamma^k_{ij}.$$

By using the second fundamental tensor b_{ij} for "raising and lowering the indices" we obtain [4, p. 232] that the $T_{ijk} = T^h_{ij}b_{hk}$ is totally symmetric and

$$(*) \qquad K_{\mathrm{II}} = H + \tfrac{1}{2}T_{ijk}T^{ijk} - (1/8K^2)\nabla_{\mathrm{II}}K.$$

Obviously, since $\frac{1}{2}T_{ijk}T^{ijk} \geqslant 0$, for a critical point P_1 of K we get

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$$(**) \qquad K_{\mathrm{II}}(P_1) \geqslant H(P_1).$$

Now we are ready to prove the following

THEOREM. *Let S be a closed, connected surface in E^3 with Gaussian curvature $K > 0$ throughout, an ovaloid in short. If $K_{\mathrm{II}} = cH^sK^r$, where c, s, r are constants and $0 < s < 1$, then S is a sphere.*

REMARK. Obviously c must be a positive constant as implied from the Gauss-Bonnet theorem.

PROOF. If we denote by dA and dA_{II} the area elements of S with respect to first and second fundamental forms, it is obvious that $dA_{\mathrm{II}} = \sqrt{K}\, dA$. Then by the Gauss-Bonnet theorem we have

$$4\pi = \int K\, dA = \int K_{\mathrm{II}}\, dA_{\mathrm{II}} = \int K_{\mathrm{II}}\sqrt{K}\, dA$$

or

$$(1) \qquad \int \sqrt{K}\left(\sqrt{K} - K_{\mathrm{II}}\right) dA = 0.$$

From the given hypothesis and equation (1) we get

$$(2) \qquad \int \sqrt{K}\,(\sqrt{K} - cH^sK^r)\, dA = 0.$$

For a critical point P_1 of K we conclude from $(**)$ that

$$cH^s(P_1)K^r(P_1) = K_{\mathrm{II}}(P_1) \geqslant H(P_1)$$

or

$$H^s(P_1)\left(cK^r(P_1) - H^{1-s}(P_1)\right) \geqslant 0$$

or

$$(3) \qquad cK^r(P_1) - H^{1-s}(P_1) \geqslant 0 \quad (\text{since } H > 0).$$

But $H^2 \geqslant K$ everywhere on S and so $H^{1-s} \geqslant K^{(1-s)/2}$ since $1 - s > 0$. Then from (3) we get

$$(4) \qquad cK^r(P_1) \geqslant K^{(1-s)/2}(P_1)$$

or

$$K^r(P_1)\left(c - K^{(1-s)/2-r}(P_1)\right) \geqslant 0$$

or

$$(5) \qquad c \geqslant K^{(1-s)/2-r}(P_1).$$

We distinguish two cases:

Case 1. Let $r < (1-s)/2$ and so $(1-s)/2 - r > 0$. In that case we choose as P_1 a point such that $K(P_1) = \sup_{P\in S} K(P)$ (such a point always exists since S is closed and K is a continuous function). Then from (5) we have

$$(6) \qquad c \geqslant K^{(1-s)/2-r} \quad \text{everywhere on } S.$$

Case 2. Let $r > (1-s)/2$ and so $(1-s)/2 - r < 0$. In that case we choose as P_1 a point such that $K(P_1) = \min_{P \in S} K(P)$. Then from (5) we have

(6′) $$c \geqslant K^{(1-s)/2-r} \quad \text{everywhere on } S.$$

Now, from (6) and (6′) we have for $r > (1-s)/2$ or $r \leqslant (1-s)/2$ that

$$c \geqslant K^{(1-s)/2-r} \quad \text{everywhere on } S$$

or

$$cH^sK^r \geqslant K^{(1-s)/2-r}H^sK^r \geqslant K^{(1-s)/2-r}K^{s/2}K^r = \sqrt{K}$$

or

(7) $$\sqrt{K} - cH^sK^r \leqslant 0.$$

From (7) we conclude that for $0 \leqslant s \leqslant 1$ it holds that

$$\sqrt{K} - cH^sK^r \leqslant 0 \text{ everywhere on } S.$$

Now, since the function $\sqrt{K} - cH^sK^r$ is nonpositive we get from (2) that

$$\sqrt{K} = cH^sK^r \quad \text{or} \quad K_{\text{II}} = cH^sK^r = \sqrt{K}.$$

But from a well-known result [3, p. 177], we conclude that S is a sphere, since $K_{\text{II}} = \sqrt{K}$. This completes the proof of the theorem.

Remark. This result contains as special cases the results in [1]–[6] for appropriate constants c, s and r.

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