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DEGENERATIONS OF CURVES IN $\mathbf{P}^3$

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ABSTRACT. In this paper we prove every connected, reduced curve in $\mathbf{P}^3$ of arithmetic genus 0, may be flatly smoothed. Moreover, we give a new example of a reduced singular curve in $\mathbf{P}^3$ which cannot be flatly smoothed.

Introduction. This paper is concerned with the following question: Given a reduced, connected curve $X \subset \mathbf{P}^3$, when is X a degeneration of a smooth curve? More precisely, given such a curve X, when may it be flatly smoothed in $\mathbf{P}^3$ to a smooth curve, that is when does it lie on an irreducible component of the Hilbert scheme of curves in $\mathbf{P}^3$ with generic member a smooth curve?

In §1, we prove if X is a connected, reduced curve in $\mathbf{P}^3$ of arithmetic genus 0, then X may always be flatly smoothed to a smooth irreducible curve.

In §2, we show there exist curves in $\mathbf{P}^3$ which cannot be flatly smoothed. An example of such a curve was given by Peskine-Szpiro in [7]. Using the techniques of determinantal schemes they showed that the reducible curve consisting of a line and a nonsingular plane elliptic curve of degree 3 intersecting in one point cannot be flatly smoothed. We give here another example using a completely different technique (namely the Riemann-Roch theorem).

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NOTATION AND TERMINOLOGY. (i) All our schemes will be projective algebraic defined over a fixed *algebraically closed* field k.

(ii) By *curve* we mean 1-dimensional scheme, by *surface* 2-dimensional scheme.

(iii) For X a scheme of dimension $n, p_a(X) = 1 - x(\mathcal{O}_X)$ where

$$x(\mathcal{O}_X) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathcal{O}_X).$$

(iv) $\mathbf{P}^n = \mathbf{P}^n_k = \operatorname{Proj} k[X_0, \ldots, X_n]$.

(v) By *flat smoothing* of a curve $X \subset \mathbf{P}^n$, we mean a flat family $\mathfrak{X} \subset \mathbf{P}^n_R$ parametrized by $T = \operatorname{Spec}(R)$, R a discrete valuation ring, with the generic fiber of $\mathfrak{X}$ smooth, the special fiber isomorphic to X. This implies that X lies on the same irreducible component of the Hilbert scheme of curves in $\mathbf{P}^n$ as a smooth curve. See [2] for definition and construction of the Hilbert scheme.

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(vi) Given $X \subset \mathbf{P}^n$ a closed subscheme, we let $\deg(X)$ = degree of X.

(vii) Given divisors D_1, D_2 on a nonsingular surface S, we let $\deg(D_1 \cdot D_2)$ be their intersection number on S.

(viii) Given a scheme X and a divisor D on X, we let $H^i(D) = H^i(X, \mathcal{O}_X(D))$. Moreover we let $h^i(D) = \dim H^i(D)$.

1. Smoothing reducible rational curves. We begin with the following lemma:

LEMMA 1.1. *Let $\mathfrak{X} \hookrightarrow \mathbf{P}^n \times T$ be a flat family of curves over T a k-scheme (where the flat map p: $\mathfrak{X} \to T$ is induced by the projection $\mathbf{P}^n \times T \to T$ and $\mathfrak{X}$ is a closed subscheme of $\mathbf{P}^n \times T$). Let $t_0 \in T$ be a closed point. Suppose the closed fiber $\mathfrak{X}_{t_0}$ is embeddable in $\mathbf{P}^m_{k(t_0)}$ $(m \leqslant n)$ by generic projection. Then locally around the closed fiber $\mathfrak{X}_{t_0}$, $\mathfrak{X}$ is embeddable in $\mathbf{P}^m \times T$.*

PROOF. Since the question is local, we may clearly assume $T = \operatorname{Spec}(\mathcal{O}_{T,t_0})$. Let M be a generic $(n - m - 1)$-dimensional linear subspace of $\mathbf{P}^n_{k(t_0)}$ disjoint from $\mathfrak{X}_{t_0}$ which defines the projection of $\mathfrak{X}_{t_0}$ into $\mathbf{P}^m_{k(t_0)}$. We claim that $\mathfrak{X} \cap (M \times T) = \varnothing$. Indeed, otherwise $\mathfrak{X} \cap (M \times T)$ would be a closed subset of $\mathfrak{X}$ and since p is proper, this intersection would be mapped to a closed subset of $T = \operatorname{Spec}(\mathcal{O}_{T,t_0})$. This then implies that $\mathfrak{X} \cap (M \times T)$ contains points in the closed fiber $\mathfrak{X}_{t_0}$ which contradicts the choice of M. Thus $M \times T$ defines a projection π: $\mathfrak{X} \to \mathbf{P}^m \times T$ and π is a finite morphism. Denote $\mathbf{P}^m \times T$ by Y and let Y_0 be the closed fiber of $Y \to T$ defined by $t_0 \in T$. Let $y \in Y_0$. Then either $\pi^{-1}(y) = \varnothing$ or $\pi^{-1}(y) = \{x\}$ (as a point set) since by hypothesis the projection defined by M is an embedding of $\mathfrak{X}_{t_0}$ into $\mathbf{P}^m_{k(t_0)}$. Then in the latter case we conclude by Nakayama's lemma that $\mathcal{O}_{Y,y} \to \mathcal{O}_{\mathfrak{X},x}$ is surjective, which means we can locally embed $\mathfrak{X}$ as required. Q.E.D.

THEOREM 1.2. *Let $X = X_1 \cup X_2$ be a curve in $\mathbf{P}^3$ with irreducible components X_1, X_2 such that $p_a(X_i) = 0$, $\deg(X_i) = d_i$, $i = 1, 2$, and X_1 and X_2 intersect in a single point, with distinct tangents at this point. Then X may be flatly smoothed in $\mathbf{P}^3$ to a smooth irreducible rational curve of degree $d_1 + d_2$.*

PROOF. We will deform over parameter scheme $T = \operatorname{Spec}(R)$, R a discrete valuation ring. Denote the fibers of $\mathbf{P}^1 \times T \to T$ by Γ_1, Γ_2, where Γ_1 is the generic fiber, and Γ_2 is the closed fiber. On the surface $\mathbf{P}^1 \times T$, choose a divisor D which intersects each Γ_i, $i = 1, 2$, in two points y_{ij}, $j = 1, 2$, with $y_{ij} \in \Gamma_i$ such that the intersection number at y_{ij} is $d_j = \deg X_j$ $(j = 1, 2)$. Thus we have $\deg(D \cdot \Gamma_i) = d_1 + d_2$ $(i = 1, 2)$.

Next, blow up the surface $\mathbf{P}^1 \times T$ at $y_{21} \in \Gamma_2$. Let $\tilde{D}$ be the proper transform of D on the blown-up surface S. Then S is fibered over T (the projection $\mathbf{P}^1 \times T \to T$ induces a morphism p: $S \to T$) with generic fiber Γ'_1 isomorphic to a projective line, and with special fiber Γ'_2 isomorphic to two projective lines joined at a single point. Note then that $\tilde{D}$ intersects the generic fiber Γ'_1 in two points, such that at one of the points $\tilde{D}$ has intersection number d_1 with Γ'_1 and at the other point $\tilde{D}$ has intersection number d_2 with Γ'_2. Moreover, $\tilde{D}$ intersects each one of the two projective lines comprising the closed fiber Γ'_2 at one point, the intersection number of $\tilde{D}$ with one of the projective lines being d_1, with the other being d_2. Therefore $\deg(\tilde{D} \cdot \Gamma'_i) = d_1 + d_2, i = 1, 2$.

Now $\tilde{D}$ induces a very ample divisor on each of the fibers Γ'_1, Γ'_2. The very ample divisor which $\tilde{D}$ induces on the generic fiber Γ'_1 defines an embedding of Γ'_1 as a smooth rational curve of degree $d_1 + d_2$ in a projective space of dimension $d_1 + d_2$, and the very ample divisor which $\tilde{D}$ induces on the closed fiber Γ'_2 defines an embedding of Γ'_2 in a projective space of dimension $d_1 + d_2$ with image two smooth rational curves joined at a single point, one curve being of degree d_1, the other being of degree d_2. We now prove that $\tilde{D}$ is very ample on S and defines an embedding of S into $\mathbf{P}_R^{d_1+d_2}$. Indeed, let t be a local parameter for R. Consider the exact sequence

$$0 \to \mathcal{O}_S(\tilde{D}) \xrightarrow{t} \mathcal{O}_S(\tilde{D}) \to \mathcal{O}_S(\tilde{D})/t\mathcal{O}_S(\tilde{D}) \to 0$$

(where by "t" we mean the map defined by multiplication by t). Then $\mathcal{O}_S(\tilde{D})/t\mathcal{O}_S(\tilde{D})$ is isomorphic to the closed fiber of $\tilde{D}$ over T. Denote this by $\tilde{D}_t$. Next consider the long exact cohomology sequence:

$$0 \to H^0(\tilde{D}) \xrightarrow{t} H^0(\tilde{D}) \to H^0(\tilde{D}_t) \to H^1(\tilde{D}) \xrightarrow{t} H^1(\tilde{D}) \to H^1(\tilde{D}_t) \to \cdots .$$

Then we want to show $H^1(\tilde{D}_t) = 0$. This follows, since $\tilde{D}_t$ is very ample and its sections give an embedding of Γ'_2 in a $d_1 + d_2$-dimensional projective space, i.e. $h^0(\tilde{D}_t) = d_1 + d_2 + 1$. But from Riemann-Roch,

$$h^0(\tilde{D}_t) - h^1(\tilde{D}_t) = 1 - p_a(\Gamma'_2) + \deg \tilde{D}_t = 1 + d_1 + d_2$$

implying $H^1(\tilde{D}_t) = 0$. Then by Nakayama's lemma we conclude $H^1(\tilde{D}) = 0$, so that $H^0(\tilde{D}) \to H^0(\tilde{D}_t)$ is surjective, i.e. we may extend sections from the closed fiber. Let s_i $(i = 0, \ldots, d_1 + d_2)$ be the sections of $\tilde{D}_t$ which define the embedding of Γ'_2. Then extend these to sections of $\tilde{D}$ and denote the extended sections again by s_i. Let $\sigma_i = \{\text{zero set of } s_i\}$ and $Z = \bigcap \sigma_i$. Then Z is a closed subset of S, and since the map $p\colon S \to T$ is proper, Z cannot intersect the generic fiber so that $Z = \varnothing$ (on the special fiber the s_i have no common zeros). Hence the s_i have no base points and so $\tilde{D}$ defines a morphism f: $S \to \mathbf{P}_R^{d_1+d_2} = Y$. We now show f is an embedding.

Let $q\colon Y \to \operatorname{Spec}(R)$ be the natural projection. Since p and q are proper, f is proper. Let Y_0 be the closed fiber of Y over $\operatorname{Spec}(R)$, and let $y \in Y_0$. Then $f^{-1}(y)$ is either empty, or one point x. Thus since the fibers are finite, and f is proper by **[1**, 4.4.2], we have that there exists a neighborhood $U_y \ni y$ such that $f\colon f^{-1}(U_y) \to U_y$ is finite. Then by Nakayama's lemma we have $\mathcal{O}_{Y,y} \to \mathcal{O}_{S,x}$ is surjective, and hence we conclude that $f|f^{-1}(U_y) \to U_y$ is an embedding. Since $\bigcup_{y \in Y_0} f^{-1}(U_y) = S$, we have $S \hookrightarrow Y = \mathbf{P}_R^{d_1+d_2}$ via the morphism f.

Finally by Lemma (1.1) we note that we can deform in $\mathbf{P}_R^3$, since the closed fiber of $S \to \mathbf{P}_R^{d_1+d_2}$ over $\operatorname{Spec}(R)$ can be embedded in projective 3-space by generic projection. Q.E.D.

COROLLARY 1.3. *Let X be a connected reduced curve in $\mathbf{P}^3$ of arithmetic genus* 0 *and degree d. Then X may be flatly smoothed to a degree d smooth irreducible rational curve.*

PROOF. We first remark that from the results of [5], it follows immediately that if X is reduced, connected with $p_a(X) = 0$, $\deg(X) = d$, then X must be a union $X_1 \cup \cdots \cup X_r$ of smooth irreducible rational curves X_i, such that $\deg(X_i) = d_i$, $d = \sum_{i=1}^r d_i$, and such that if X_i and X_j intersect, then they intersect at precisely one point, with distinct tangents at this point. Moreover, X cannot have any "loops", i.e. there exists no sequence $(i_1, \ldots, i_k)$ $(k > 1)$ such that $X_{i_1} \cap X_{i_2} \neq \varnothing, \ldots, X_{i_{k-1}} \cap X_{i_k} \neq \varnothing, X_{i_k} \cap X_{i_1} \neq \varnothing$. Then from the above description of X, X must be isomorphic to a curve constructed in the following manner: Blow up $\mathbf{P}^1$ a finite number of times to get a curve Y with irreducible components $Y_1, \ldots, Y_r$ $(Y_i \approx \mathbf{P}^1)$ such that $Y_i \cap Y_j \neq \varnothing$ if and only if $X_i \cap X_j \neq \varnothing$. Next embed Y into $\mathbf{P}^d = \mathbf{P}^{d_1 + \cdots + d_r}$ via the linear system $|\sum_{i=1}^r d_i P_i|$ $(P_i \in Y_i, P_i \notin Y_j, i \neq j)$, and finally project generically the image of Y in $\mathbf{P}^d$ into $\mathbf{P}^3$. If we call the resulting curve Z, then X will be isomorphic to a curve Z constructed as above.

Finally given the above construction of X as a blowing up of $\mathbf{P}^1$, recalling the proof of Theorem 1.2, it is clear we may use an identical method (of blowing up the closed fiber of $\mathbf{P}^1 \times T \to T$ a finite number of times and choosing a divisor with the proper intersection properties, etc.) to show X has a flat smoothing to a degree d smooth rational curve in $\mathbf{P}^3$. Q.E.D.

2. A reduced curve with no flat smoothing. We now show that not every reduced curve in $\mathbf{P}^3$ can be flatly smoothed. We begin with the following classical result which appears in both [4] and [6].

PROPOSITION 2.1. *There exist no smooth, irreducible projective curves of degree* 5, *arithmetic genus* 3.

PROOF. Suppose to the contrary $X \subset \mathbf{P}^n$ were smooth irreducible of degree 5, genus 3. Then there would exist a very ample divisor D on X such that $\deg D = 5$. But by Riemann-Roch,

$$h^0(D) - h^1(D) = 1 - p_a(X) + \deg D = 3.$$

Now since $\deg D = 5 > 2p_a(X) - 2 = 4$, we have $h^1(D) = 0$, which implies $h^0(D) = 3$. Then since D is very ample, we see that D defines an embedding of X as a plane curve. But every smooth plane curve of degree 5 has genus 6. Q.E.D.

EXAMPLE 2.2. We now give our example of a reduced curve in $\mathbf{P}^3$ with no flat smoothing. Let $X = X_1 \cup X_2 \subset \mathbf{P}^3$ be such that X_1 is a smooth degree 4 plane curve, and X_2 is a line lying in a different plane, and such that X_1 and X_2 intersect in precisely one point. Then X has degree 5 and arithmetic genus 3. Now since degree and arithmetic genus are preserved by flat deformation (see [3]), we see by Proposition (2.1) that X cannot be flatly smoothed to an irreducible smooth curve.

Finally, to show that X cannot be flatly smoothed to a smooth reducible curve in $\mathbf{P}^3$, we need only show that there are no smooth reducible curves of degree 5, arithmetic genus 3. We have the following two possibilities for Y a smooth reducible curve of degree 5:

Case (i). $Y = Y_1 \cup Y_2$, Y_1 a line, Y_2 a smooth degree 4 curve (perhaps reducible). Then the maximum possible arithmetic genus for Y_2 is 3 (in which case Y_2 is a plane curve). Therefore since Y_1 and Y_2 do not intersect, $p_a(Y) \leqslant 2$.

Case (ii). $Y = Y_1 \cup Y_2$, Y_1 a smooth irreducible conic, Y_2 a smooth degree 3 curve (perhaps reducible). Then $p_a(Y_1) = 0$ and $p_a(Y_2) \leqslant 1$ (the maximum arithmetic genus for a smooth degree 3 curve is 1, the plane curve case). Then $p_a(Y) \leqslant 0$, since Y_1 and Y_2 do not intersect.

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