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INJECTIVE NEAR-RING MODULES OVER Z_n

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ABSTRACT. In this note, we show that an injective unital near-ring module over Z_n has trivial centre if $n > 3$. As a consequence there are no injective unital near-ring modules over Z_3.

The question of the existence of injective modules over near-rings has received a lot of attention since the publication of Seth and Tewari [7] sparked off a series of counter-examples to some of their results. Banaschewski and Nelson [1], Mason [3], Oswald [4] and Prehn [6] have all shown that in general injective modules over a near-ring do not exist, not even in the category of unital near-ring modules over Z. But Banaschewski and Nelson [1] and Prehn [6] have shown that, under certain very special circumstances injective near-ring modules can exist. The two cases are very different, but they both use special properties of the near-ring concerned to return to abelian modules. This leaves open the question: for which near-rings do injective modules exist? The probable answer is not many! In the present contribution, we show that injective unital near-ring modules over Z_n have trivial centre, and hence that the ideas of Banaschewski and Nelson and Prehn do not help in this case. For basic definitions and results, see Pilz [5], the only difference being that we use left rather than right near-rings. Here Z_n is the ring of integers modulo n, so the distinction between right and left near-rings is, in fact, irrelevant in this context.

We first note that the category of unital near-ring modules over Z_n is just the category of groups of exponent n. Contrary to the usual practice when dealing with near-ring modules, we will write all groups multiplicatively. The main result follows immediately.

THEOREM. *An injective unital near-ring module over Z_n has trivial centre, if $n \geqslant 3$.*

PROOF. We prove this result by contradiction. Let M be an injective module over Z_n with nontrivial centre A. Let p be a prime dividing n such that A contains an element of order p, and let $n = pq$ (q may be 1). Let C be a cyclic group of order p, generated by c. Let $D = \prod_{i=1}^{p} M_i$, where $M_i \cong M$ and $\prod$ indicates direct product. Then $f \in D = M^C$ can be thought of as a

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function from C to M, where we write $f(i)$ for $f(c^i)$ and $f(i) \in M_i$. Define c as an automorphism of D by $f^c(i) = f(i-1)$. Then $D \cdot C$, the semidirect product of D by C is the wreath product of M by C (see Hall [2]).

Let $B \subseteq D$ be defined by $B = \{f \in D;\ f(i) = ma_i,\ m \in M,\ a_i \in A$ for $1 \leqslant i \leqslant p$ and $a = a_1 \cdots a_p$ has order dividing $q\}$. We show that B is a subgroup of D normalised by C. Certainly B is nonempty. Let $f, g \in B$. Then

$$fg^{-1}(i) = ma_i(nb_i)^{-1} = mn^{-1}a_ib_i^{-1}$$

and $ab^{-1} = a_1 \cdots a_p(b_1 \cdots b_p)^{-1}$ has order dividing q since A is an abelian group. Here $g(i) = nb_i$, $n \in M$, $b_i \in A$ and $b = b_1 \cdots b_p$ has order dividing q. Finally $f^c(i) = f(i-1) = ma_{i-1}$ and this obviously lies in B. So $B \cdot C$ is a subgroup of $D \cdot C$. Note that $f(1) \cdots f(p) = m^pa_1 \cdots a_p$, since $a_i \in A$ for $1 \leqslant i \leqslant p$, and this element has order dividing q.

We next show that BC is a group of exponent n. Let $fc^r \in BC$. Then

$$(fc^r)^p = fc^r \cdots fc^r = ff^{c^{-r}}f^{c^{-2r}} \cdots f^{c^{-(p-1)r}}c^{pr} = h$$

where $h \in B$, $h(i) = f(i)f(i+r) \cdots f(i+(p-1)r)$. If $r = 0$, then

$$f^n(i) = (ma_i)^n = m^na_i^n = e$$

the identity. If $1 \leqslant r \leqslant p-1$, then r and p are coprime and so $h(i) = f(1) \cdots f(p)$. By the remark above $h(i)$ has order dividing q, so $h^q = e$, $(fc^r)^n = (fc^r)^{pq} = e$.

Embed M in BC by $m \to \bar{m}$, where $\bar{m}(i) = m$ for $1 \leqslant i \leqslant p$. Since M is injective, $BC = MN$ is the semidirect product of M and N, and N is normal in BC. As $c \in MN$, we must have $c = \bar{m}^{-1}\overline{mc}$ for some $\bar{m}$, where $\overline{mc} \in N$. There are two cases now.

(i) We can choose $p \geqslant 3$. Let $e \neq a \in A$ have order p and define $g \in B$ by $g(1) = a$, $g(2) = a^{-1}$, $g(i) = e$ for $2 < i \leqslant p$. Then $[g, \overline{mc}] \in N$. But g commutes with $\bar{m}$, so $[g, \overline{mc}] = [g, c] \in N$. Let $h = [g, c]$. Then

$$h(i) = g^{-1}(i)g^c(i) = g^{-1}(i)g(i-1).$$

Define $k = \prod_{i=1}^{p-1} h^{x(i)}$ where $x(i) = \frac{1}{2}i(i+1)c^{i-1}$. From the definition of h, we have $h(i) \neq e$ only if $i = 1$, 2 or 3, and $h(1) = a^{-1}$, $h(2) = a^2$, $h(3) = a^{-1}$, since $p \geqslant 3$. Hence

$$k(i) = a^{-1} \quad \text{as } -\tfrac{1}{2}(i-2)(i-1) + i(i-1) - \tfrac{1}{2}i(i+1) = -1 \qquad \text{for } 3 \leqslant i \leqslant p-1,$$

$$k(1) = a^{-1} \quad \text{as } -1 - \tfrac{1}{2}p(p-1) \equiv -1 \mod p;$$

$$k(2) = a^{-1} \quad \text{as } 2 - \tfrac{1}{2}2 \cdot 3 = -1,$$

$$k(p) = a^{-1} \quad \text{as } -\tfrac{1}{2}(p-2)(p-1) + p(p-1) \equiv -1 \mod p.$$

Hence $e \neq k \in M \cap N$, a contradiction.

(ii) We cannot choose $p \geqslant 3$. Then $n = 2^r$ and $r \geqslant 2$. So $p = 2$, $2|q$ and we can choose $e \neq a \in A$ of order 2. Define $g \in B$ by $g(1) = a$, $g(2) = e$. Then $g \in B$ as $2|q$. As before $[g, c] \in N$. Write $h = [g, c]$. Then

$$h(1) = g(1)^{-1}g(2) = a^{-1} = a, \qquad h(2) = g(2)^{-1}g(1) = a.$$

Hence $e \neq h \in M \cap N$, another contradiction.

This finishes the proof of the theorem.

COROLLARY. *There are no injective unital near-ring modules over Z_3.*

PROOF. This follows from the theorem and the fact that all groups of exponent 3 are nilpotent and so have nontrivial centre. (See Hall **[2]**.)

Note that groups of exponent 2 are necessarily abelian and that unital near-ring modules over Z_2 are vector spaces over Z_2, and for these injectives exist in abundance.

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