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THE MODULAR GROUP-RING OF A FINITE p-GROUP

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ABSTRACT. For a finite p-group G and the field k of p elements, we investigate the embedding of G in the group G^* of elements of the group-ring kG having coefficient-sum equal to 1. Of particular interest is the question of when G has a normal complement in G^*, for in this case simple proofs can be given for a number of diverse known results.

Since its inception in [5], the study of modular group-rings of finite p-groups has largely centred around the problem (see [1], [3], for example): "When does isomorphism of group-rings imply isomorphism of groups?" A key role in these investigations has been played by the group of units in the group-ring.

Fixing our notation, we let p denote a fixed prime, G a finite p-group, k the field of p elements and kG the group-ring of G over k. The group of units of kG is simply $kG \setminus U$, where U is the augmentation ideal of kG. Furthermore, $kG \setminus U = G^* \times k^*$, where $G^* = 1 + U$, and $k^* = k \setminus \{0\}$. We call G^* the mod p envelope of G, and denote the embedding $G \rightarrowtail G^*$ by ι_G, or simply ι. Note that G^* is also a p-group.

Some properties of ι are as follows:

$$\mathbf{Z}(G) = \mathbf{Z}(G^*) \cap G, \tag{1}$$

$$\mathbf{N}_{G^*}(G) = G\mathbf{C}_{G^*}(G), \tag{2}$$

$$G' = (G^*)' \cap G, \tag{3}$$

$$\Phi(G) = \Phi(G^*) \cap G. \tag{4}$$

None of these is very hard; (2) and (3) are proved in [2], and (4) in [6]. We have not yet, however, found a proof for

$$G^p = (G^*)^p \cap G. \tag{5}$$

Incidently, the truth of (2) was established independently by the present author in an (entirely abortive) attempt to find a simple proof of Gaschutz' theorem [4].

We now define the class $\mathfrak{L}_p$ of finite p-groups to consist of those G which have a normal complement in G^*. The properties (1)–(5), along with many others, are immediately obvious for groups in $\mathfrak{L}_p$, and it only remains to

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establish the extent of this class. The purpose of this article is to outline the present state of our knowledge about this problem.

The embedding $\iota: G \rightarrowtail G^*$ gives rise to a number of other questions which are beyond the scope of this article. For instance, what (if anything) can be said about the direct limit of the sequence $G \subseteq G^* \subseteq G^{**} \subseteq \cdots$? Can the embedding $G \subseteq G^*$ be defined by any useful universal property? Are there any significant differences in the theory when k is replaced by an arbitrary field of characteristic p? Can we find a formula relating $\exp G$ and $\exp G^*$? While these are equal for abelian groups, equality does not hold in general, as the example of the elementary nonabelian group of order p^3 shows (see [6]).

It would also be of interest to investigate the relations between the modular representation theory (and also the cohomology) of G and G^*, for if $E(G)$ denotes the representation algebra of G (the vector space over the complex numbers with isomorphism classes of indecomposable kG-modules as a basis, and multiplication the Kronecker product of representations), then the definition of G^* ensures that the restriction homomorphism: $E(G^*) \to E(G)$ splits.

Finally, and perhaps hardest of all, is there an algorithm for finding a presentation for G^* (in terms of generators and relations) given a presentation of G?

Theorem 1. *If G is cyclic, then $G \in \mathfrak{L}_p$.*

Proof. There is an elementary result, which may be thought of as a lemma for the basis theorem or as a consequence of it, to the effect that for any finite abelian p-group G and any $x \in G$ with $|x| = \exp G$, $\langle x \rangle$ is a direct factor of G. This, together with the obvious remark that $\exp G = \exp G^*$ when G is abelian, proves the result.

We next give a simpler proof of a result in [6].

Theorem 2. *If $G, H \in \mathfrak{L}_p$, then $G \times H \in \mathfrak{L}_p$.*

Proof. The epimorphisms from $G \times H$ to G and H induce epimorphisms

$$(G \times H)^* \begin{array}{l} \xrightarrow{\nu_1} G^* \\ \xrightarrow{\nu_2} H^* \end{array}$$

If N_1 and N_2 are normal complements (which exist by hypothesis) for G and H in G^* and H^*, respectively, then let $\bar{N}_1$ and $\bar{N}_2$ denote their pre-images in $(G \times H)^*$ under ν_1 and ν_2 respectively. $\bar{N}_1$ and $\bar{N}_2$ are clearly normal subgroups of $(G \times H)^*$ such that

$$\bar{N}_1 \cap (G \times H) = H, \qquad \bar{N}_2 \cap (G \times H) = G,$$

so that if we let $N = \bar{N}_1 \cap \bar{N}_2$, then $N \cap (G \times H) = E$. Furthermore,

$$|(G \times H)^*: N| \leqslant |(G \times H)^*: \bar{N}_1|\,|(G \times H)^*: \bar{N}_2| = |G^*: N_1|\,|H^*: N_2| = |G|\,|H| = |G \times H|.$$

Thus N is the required normal complement.

THEOREM 3. *If G is abelian, then $G \in \mathfrak{L}_p$.*

PROOF. An immdiate consequence of Theorems 1 and 2.

The next result is due to Tench **[8]**, and yields the converse of Theorem 2.

THEOREM 4. *If G belongs to $\mathfrak{L}_p$, then so does any normally complemented subgroup H of G.*

PROOF. Let α: $H^* \to G^*$ be the inclusion induced by $H \leqslant G$, let β: $G^* \to G$ be a splitting for ι_G, and let γ: $G \to H$ be a splitting for $H \leqslant G$. Then the composite

$$H^* \xrightarrow{\alpha} G^* \xrightarrow{\beta} G \xrightarrow{\gamma} H$$

is clearly a splitting for ι_H, whose kernel is thus the required normal complement.

THEOREM 5. *For any G, $G^* \in \mathfrak{L}_p$.*

PROOF. Note that G^*, being a subset of kG, is closed under the formation of linear combinations of its elements, provided the coefficient-sum is equal to 1. Now any element of G^{**} is just a "formal" linear combination of this type, and thus gives rise to a unique element of G^* (the corresponding "real" linear combination). This mapping is easily seen to be an epimorphism from G^{**} to G^* which fixes G^* elementwise. It is thus a splitting for ι_{G^*} and its kernel is the required normal complement.

THEOREM 6. *If G_n denotes the Sylow p-subgroup of $GL(n,p)$, then $G_n \in \mathfrak{L}_p$ for all n.*

PROOF. The embedding $G_n \to GL(n,p) \to M_n(k)$ extends by linearity to a homomorphism of rings $kG_n \to M_n(k)$ whose restriction to G_n^* is such that all its images are units in $M_n(k)$. We thus obtain a homomorphism α: $G_n^* \to GL(n,p)$ which fixes G_n elementwise. But since G_n^* is a p-group and G_n is a Sylow p-subgroup of $GL(n,p)$, we must have $\operatorname{Im} \alpha = G_n$, so that $\operatorname{Ker} \alpha$ forms the required normal complement.

Note. By examining certain sets of upper triangular matrices with 1's on the main diagonal, this argument can be extended to show that various other p-subgroups of $GL(n,p)$ lie in $\mathfrak{L}_p$. Note further that if we knew $\mathfrak{L}_p$ to be subgroup-closed, it would follow from Theorem 6 (or from Theorem 5) that $\mathfrak{L}_p$ contained all finite p-groups.

Finally, for the sake of completeness, we list a few groups of small order in $\mathfrak{L}_p$ (see [7] for the proof).

THEOREM 7. *The following groups belong to $\mathfrak{L}_p$:*

(a) *the two nonabelian groups of order* 8 ($p = 2$),

(b) *the nonabelian group of exponent p and order* p^3 ($p > 2$),

(c) *the three nonabelian indecomposable groups of exponent* 4 *and order* 16 ($p = 2$).

Note in conclusion that the "smallest" group not definitely known to belong to $\mathfrak{L}_p$ is the dihedral group D_{16} of order 16 ($p = 2$). A programme involving the conjugacy classes of D_{16}^* is currently in preparation to decide the question using a high-speed computing machine.

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