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ON INTERPOLATING FUNCTIONS WITH MINIMAL NORM

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ABSTRACT. Let H^∞ denote the Banach algebra of bounded analytic functions in the unit disc $\{z: |z| < 1\}$. If f is an extreme point in the unit ball of H^∞, there is always a Blaschke product B, whose zeros form an interpolating sequence tending to one point of the unit circle, such that $\|f + Bh\|_\infty > 1$ if $h \in H^\infty$ and $h \neq 0$. An application of this result to the theory of best approximation is given.

Let H^∞ denote the Banach algebra of all bounded analytic functions in the unit disc D. A sequence $S = \{z_\nu\} \subset D$ is the zero set of a nonzero function in H^∞ if and only if $\Sigma_\nu 1 - |z_\nu| < \infty$. The Blaschke product corresponding to S is given by

$$B(z) = \prod_\nu \frac{|z_\nu|}{-z_\nu} \frac{z - z_\nu}{1 - \bar{z}_\nu z}$$

if $z_\nu \neq 0$, $\nu = 1, 2, \ldots$. It is well known that $B \in H^\infty$ and that B has norm one and $S = B^{-1}(0)$. We also remark that B is analytic at each $z_0 \in T = \{z: |z| = 1\}$ not in the closure of $\{z_\nu\}$.

If the restriction map $f \to \{f(z_\nu)\}_\nu$, maps H^∞ onto l^∞, S is called an interpolating sequence. For more details about H^∞ and interpolating sequences, we refer to [4].

In this note we prove the following result.

THEOREM 1. *If h is an extreme point in the unit ball of H^∞, there exists an interpolating sequence $\{z_\nu\}$ tending to one point, such that $\|f\| > \|h\|$ if $f \in H^\infty, f \neq h$, and $f(z_\nu) = h(z_\nu)$, $\nu = 1, 2, \ldots$.*

REMARK. This result, which extends recent work announced by D. S. Jerison [5], has also been proved independently by D. Marshall using similar ideas.

There is a consequence in the theory of best approximation by H^∞ functions which we would like to point out before giving the proof of Theorem 1.

Let z_0 denote the clusterpoint of $\{z_\nu\}$ on T. We assume that h does not extend continuously to any point on T. Let $f_1 \in H^\infty$ have analytic continuation across $T \setminus \{z_0\}$ and satisfy $f_1(z_\nu) = h(z_\nu)$, $\nu = 1, 2, \ldots$. Such a function exists [3].

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If B denotes the Blaschke product corresponding to $\{z_\nu\}$, we can write $h = f_1 + Bf$, where $f \in H^\infty$ is the unique best H^∞-approximation to f_1/B on T. (If $g \in H^\infty$, we assume it is extended to T by taking the radial limits $g^*(e^{i\theta}) = \lim_{r\to 1} g(re^{i\theta})$.) So even if f_1/B extends to be analytic near $T \setminus \{z_0\}$, f may be discontinuous at each point of $T \setminus \{z_0\}$. This should be compared with recent work of L. Carleson and S. Jacobs [1].

Proof of Theorem 1. If $g \in H^\infty$, g^* will always denote its radial limits on T, and $\|g^*\|_\infty$ denotes the norm of g^* in $L^\infty(d\theta)$, where $d\theta$ denotes Lebesgue measure on T. It is well known [4] that g is an extreme point in H^∞ if and only if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1 - |g^*|)\, d\theta = -\infty. \tag{1}$$

Since h is extreme, there is a number θ_0 such that $\int_{\theta_0 - t}^{\theta_0 + t} \log(1 - |h^*|)\, d\theta = -\infty$ for any $t > 0$. We may assume $\theta_0 = 0$. The idea of the proof is as follows: We choose an interpolating sequence $\{z_\nu\}$ tending to 1 such that $|h(z_\nu)| \to 1$ and with the following property:

If $f \in K = \{g \in H^\infty: \|g\| = 1, g(z_\nu) = h(z_\nu), \nu = 1, 2, \ldots\}$, then f is an extreme point in the unit ball of H^∞. Since $h \in K$ and K is convex, this implies that $K = \{h\}$.

To find $\{z_\nu\}$, let $I_n = \{z \in T: 2(n+1)^{-1} < |z - 1| \leqslant 2n^{-1}\}$ for $n = 1, 2, \ldots$. Assume first there are constants M_n such that $|h^*| \leqslant M_n < 1$ on I_n, $n = 1, 2, \ldots$. We choose compact sets $K_n \subset I_n$, $n = 1, 2, \ldots$, with the following two properties for each n:

$$\int_{K_n} \log(1 - |h^*|)\, d\theta \leqslant 2^{-1}\int_{I_n} \log(1 - |h^*|)\, d\theta; \tag{2}$$

$$\lim_{r\to 1} f_r(e^{i\theta}) = f^*(e^{i\theta}) \text{ exists uniformly for } e^{i\theta} \in K_n. \tag{3}$$

Fix n, and subdivide I_n into a collection of arcs $\{I_{n,k}\}_k$ of equal length δ_n which is so small that

$$|h^*(e^{i\theta}) - h(re^{i\theta})| < 1 - M_n \tag{4}$$

if $e^{i\theta} \in K_n$ and $r \geqslant 1 - 2\delta_n$.

From $\{I_{n,k}\}_k$ we choose a subcollection $\{J_{n,k}\}_k$ such that $J_{n,k} \cap K_n \neq \varnothing$ for each k, and with the following two properties:

$$\sum_k \int_{K_n \cap J_{n,k}} \log(1 - |h^*|)\, d\theta \leqslant 4^{-1}\int_{K_n} \log(1 - |h^*|)\, d\theta. \tag{i}$$

(ii) No arc of length less than $2\delta_n$ on T can intersect to different arcs from $\{J_{n,k}\}_k$.

Let ess $\sup\{|h^*(e^{i\theta})|, e^{i\theta} \in K_n \cap J_{n,k}\} = 1 - \eta_{nk}$. Then we get from (4), and since $1 - \eta_{nk} \leqslant M_n$, that there is a point $\zeta_{nk} \in J_{n,k}$ such that if $z_{nk} = (1 - 2\delta_n)\zeta_{nk}$, then

$$|h(z_{nk})| \geqslant 1 - 2\eta_{nk} - (1 - M_n) \geqslant 1 - 3\eta_{nk}.$$

Let $\tilde{J}_{n,k}$ be the arc centered at $\zeta_{n,k}$ and with length $2\delta_n$. By (ii), $\{\tilde{J}_{n,k}\}_k$ is a disjoint collection of arcs. If $S_n = \{z_{n,k}\}_k$, we define our sequence $\{z_\nu\}_\nu$ as the union of all S_n, $n = 1, 2, \ldots$.

We now make the following claim:

If $f \in H^\infty$, $\|f\| < 1$, and $|f(z_\nu)| \geq |h(z_\nu)|$, $\nu = 1, 2, \ldots,$ then $\int_{-\pi}^{\pi} \log(1 - |f^*|)\, d\theta = -\infty$.

To prove the claim, we use that if $|f(z_{n,k})| \geq |h(z_{nk})| \geq 1 - 3\eta_{nk}$, then there is a set $E_{nk} \subset \tilde{J}_{n,k}$ and absolute constants A_1 and A_2 such that

$$|f^*| \geq 1 - A_1\eta_{nk} \quad \text{on } E_{nk} \tag{5}$$

and

$$\begin{aligned} \text{length } E_{nk} &\geq A_2 \text{ length } \tilde{J}_{n,k} = 2A_2\delta_n \\ &= 2A_2 \text{ length } J_{n,k}. \end{aligned} \tag{6}$$

The existence of A_1 and A_2 comes out of estimating the Poisson kernel for $z_{n,k}$ on the arc $\tilde{J}_{n,k}$.

Using (5), (6) and the definition of $\eta_{n,k}$, it is easy to see that

$$\begin{aligned} &\int_{E_{nk}} \log(1 - |f^*|)\, d\theta \\ &\quad \leq 2A_2 \int_{K_n \cap J_{n,k}} (\log(1 - h|h^*|) + \log A_1)\, d\theta. \end{aligned}$$

If we sum these inequalities for each $J_{n,k} \in \{J_{n,k}\}_k$, and then for $n = 1, 2, \ldots,$ we get, using property (ii) about $\{J_{n,k}\}_k$ and (2), that

$$\begin{aligned} \int_T \log(1 - |f^*|)\, d\theta &\leq \sum_n \sum_k \int_{E_{nk}} (\log 1 - |f^*|)\, d\theta \\ &\leq 4^{-1}A_2 \int_T (\log 1 - |h^*| + \log A_1)\, d\theta = -\infty, \end{aligned}$$

which proves the claim.

We verify now that $\{z_\nu\}_\nu$ is an interpolating sequence. By a characterization of such sequences essentially due to L. Carleson [**2**, Lemma 1], we need only verify the following two inequalities:

$$\sum_{z_\nu \in R(\theta, h)} (1 - |z_\nu|) \leq C \cdot h \tag{a}$$

(where $R(\theta, h) = \{z: \theta \leq \arg z \leq \theta + h,\ 0 < 1 - |z| \leq h\}$ and C is independent of h), and

$$\inf_{\mu \neq \nu} \left| \frac{z_\nu - z_\mu}{1 - \bar{z}_\mu z_\nu} \right| \geq \delta \tag{b}$$

for some $\delta > 0$ and $\nu = 1, 2, \ldots$.

By construction there is no z_μ with $\mu \neq \nu$ in the sector

$$S_\nu = \left\{z: \arg z_\nu - \tfrac{1}{2}(1 - |z_\nu|) < \arg z < \arg z_\nu + \tfrac{1}{2}(1 - |z_\nu|)\right\}.$$

It is easy to verify that this fact implies both (a) and (b) above. In fact, the arc $\tilde{J}_\nu = T \cap \partial S_\nu$, corresponding to z_ν, has length $1 - |z_\nu|$, and since the collection $\{J_\nu\}_\nu$ is disjoint, (a) must hold with $c = 2$. Elementary geometric considerations also show that the disc $D_\nu = \{z: |(z - z_\nu)/(1 - z_\nu z)| < \delta\}$ is contained in S_ν if δ is sufficiently small, but independent of ν.

The proof is complete except that we assumed $|h^*| \leqslant M_n < 1$ on each I_n. If this is not the case, we divide h by a suitable outer function F such that $h_1 = h \cdot F^{-1}$ is an extreme point satisfying all our hypothesis.

We can arrange it so that $|F(z)| > 1$, $z \in D$. If the above claim is proved for h_1, it is certainly also true for h.

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