PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 1, January 1978

APPROXIMATION BY POLYNOMIALS IN z AND ANOTHER FUNCTION[1]

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ABSTRACT. We present some progress in the understanding of when and why polynomials in z and a given continuous function f are uniformly dense in all continuous complex valued functions on the closed unit disk. The first theorem requires that f be an ACL^2-function in a neighborhood of the disk which satisfies $\operatorname{Re} f_{\bar z} \geqslant |f_z|$ almost everywhere and $f^{-1}(f(a))$ is countable for each a. The second theorem requires that f have a special form and satisfy $|f_{\bar z}| > |f_z|$ everywhere except at the origin. The form is that $f = \bar z^k \phi(|z|^{2k})$ where ϕ is a complex valued function of a real variable satisfying ϕ is continuous in $[0, 1]$, ϕ' exists in $(0, 1)$ and where k is a positive integer.

1. Introduction. Denote by $\mathbf{C}$ the complex plane and by z the identity function. D denotes the closed disk $\{s \in \mathbf{C}: |s| \leqslant 1\}$, $C(D)$ will be the set of all complex valued continuous functions on D and for f in $C(D)$, P_f will be the uniform closure on D of all finite sums $\sum_{i,j \geqslant 0} a_{ij} z^i f^j$ where $a_{ij} \in \mathbf{C}$. If $f = U + iV$, then the Jacobian determinant of f is given by $\det\begin{pmatrix} U_x & U_y \\ V_x & V_y \end{pmatrix}$. Alternately, if we put $f_z = (U_x + V_y)/2 + i(V_x - U_y)/2$ and $f_{\bar z} = (U_x - V_y)/2 + i(V_x + U_y)/2$, then direct computation yields $U_x V_y - U_y V_x = |f_z|^2 - |f_{\bar z}|^2$. In particular, f has negative Jacobian determinant if and only if $|f_{\bar z}| > |f_z|$. Finally, Lebesgue two dimensional measure in $\mathbf{C}$ is denoted by m.

We are concerned with the following open question. Assume f is smooth in a neighborhood of D and $|f_{\bar z}| > |f_z|$; is $P_f = C(D)$? Our main frame of reference consists of two papers ([6] and [7]) of J. Wermer and two papers ([3] and [4]) by the author. In [3], it is shown that $|f_{\bar z}| > |f_z|$ implies all rational functions in z and f, which are finite, are uniformly dense in $C(D)$ and in [4] it is proved that if $\operatorname{Re} f_{\bar z} \geqslant |f_z|$ everywhere in the interior of D, then $P_f = C(D)$.

In §2, we prove the following theorem and a mild extension of it.

THEOREM I. *Let f be an ACL^2-function in a neighborhood of D which satisfies*
(i) $\operatorname{Re} f_{\bar z} \geqslant |f_z|$ *a.e. in D and*
(ii) $f^{-1}(f(a))$ *is countable for each a in D.*
Then $P_f = C(D)$.

In §3, we prove the following theorem.

Received by the editors July 18, 1977.
AMS (MOS) subject classifications (1970). Primary 46J10, 56J15; Secondary 30A60, 30A82.
[1]This work was partially done while the author was at Boston College on a faculty fellowship.

THEOREM II. *Assume* ϕ *is a complex valued function of a real variable satisfying* ϕ *is continuous in* $[0, 1]$ *and* ϕ' *exists in* $(0, 1)$. *Put* $f = \bar{z}^k\phi(|z|^{2k})$ *where* k *is a fixed but arbitrary positive integer. If* $|f_{\bar{z}}| > |f_z|$ *everywhere in* $\operatorname{Int} D - \{0\}$, *then* $P_f = C(D)$.

The author is grateful to Bruce Palka for helpful suggestions concerning the main theorem in §2, and to Andrew Browder for helpful conversations about §§ 2 and 3.

2. Before we prove Theorem I, we quote some needed definitions and results which may be found in [5].

Let Q be a closed rectangle in $\mathbf{C}$. A complex valued function f on Q is said to be ACL or absolutely continuous on lines if f is continuous in Q and if f is absolutely continuous on almost every line segment in Q parallel to the coordinate axes. A function f on an open set U in $\mathbf{C}$ is ACL when f restricted to any rectangle in U is ACL. An ACL-function on an open set U has partial derivatives a.e. in U and they are Borel functions. When the partial derivatives of an ACL-function f are locally L^2-integrable, f is said to be ACL^2. (Reference: §26 in [5].)

DEFINITION. Given a family Γ of locally rectifiable paths in $\mathbf{C}$, let $F(\Gamma)$ denote all Borel functions $\rho\colon \mathbf{C} \to [0, \infty]$ such that $\int_\gamma p\, ds \geqslant 1$ for each γ in Γ. The modulus of Γ, denoted by $M(\Gamma)$, is the number $\inf_{\rho\in F(\Gamma)} \int_{\mathbf{C}} \rho^2\, dm$; if $F(\Gamma) = \varnothing$, then $M(\Gamma) = \infty$.

THEOREM 2.1. $M(\Gamma)$ *has the following properties*:
(i) $M(\Gamma)$ *is invariant under similarity mappings*,
(ii) $M(\varnothing) = 0$,
(iii) *if* $\Gamma_1 \subset \Gamma_2$, *then* $M(\Gamma_1) \leqslant M(\Gamma_2)$,
(iv) $M(\bigcup_{i=1}^{\infty} \Gamma_i) \leqslant \sum_{i=1}^{\infty} M(\Gamma_i)$,
(v) *if each* γ *in* Γ_2 *has a subcurve which belongs to* Γ_1, *then* $M(\Gamma_2) \leqslant M(\Gamma_1)$.

For a proof, see §6 in [5].

PROPOSITION 2.2. *Let* $0 < \varepsilon < R < \infty$ *and let* $S(\varepsilon)$, $S(R)$ *equal respectively the circles about* 0 *with radii* ε, R. *Let* Y *be a measurable set on the unit circle and take* $\Gamma =$ *all radial line segments* γ *joining* $S(\varepsilon)$ *to* $S(R)$ *such that* $|\gamma| \subset \{s \in \mathbf{C}\colon s/|s| \in Y\}$. *Then* $M(\Gamma) = \sigma(Y)(\operatorname{Log} R/\varepsilon)^{-1}$ *where* $\sigma(Y) = 2(m\{ru\colon 0 < r < 1 \text{ and } u \in Y\})$.

In order to prove the proposition, one can mildly modify the argument given in Example 7.5 of [5].

THEOREM 2.3 (FUGLEDE). *Suppose* U *is an open set in* $\mathbf{C}$ *and that* f *is an* ACL^2-*function on* U. *Let* T *be the family of all locally rectifiable paths in* U *which have a closed subpath on which* f *is not absolutely continuous. Then* $M(T) = 0$.

PROOF OF THEOREM I. It suffices to show that for s, t in D we have $\operatorname{Re}[(t - s)(f(t) - f(s))] \geqslant 0$. That $P_f = C(D)$ will follow from this fact and from the hypothesis that $f^{-1}(f(a))$ is countable for each a exactly as in Theorem A of [4]. Also, the continuity of f allows us to restrict our attention to the situation $t \in D$, $s \in \operatorname{Int} D$ and $t \neq s$. Fix such an s and t. Choose $\varepsilon > 0$ so small that $t \notin D(s; \varepsilon) = \{w: |w - s| \leqslant \varepsilon\} \subset \operatorname{Int} D$ and let $s_\varepsilon = s + \varepsilon(t - s)/|t - s|$. Again, by the continuity of f, it suffices to prove that $\operatorname{Re}[(t - s_\varepsilon)(f(t) - f(s_\varepsilon))] \geqslant 0$ and this is what we will show.

Choose T as in Theorem 2.3; there are two cases to consider.

Case I. The line segment ν joining s_ε to t is not in T and $\operatorname{Re} f_{\bar z} \geqslant |f_z|$ a.e. with respect to linear measure on ν.

By Theorem 2.3

$$f(t) - f(s_\varepsilon) = \int_\nu (f_x dx + f_y dy)$$

and so

$$\operatorname{Re}\big[(t - s_\varepsilon)(f(t) - f(s_\varepsilon))\big] = \int_\nu \operatorname{Re}\big[(t - s_\varepsilon) df(t - s_\varepsilon)\big].$$

A straightforward calculation yields

$$\operatorname{Re}\big[(t - s_\varepsilon) df(t - s_\varepsilon)\big] = \operatorname{Re}\big[(t - s_\varepsilon)^2 f_z + |t - s_\varepsilon|^2 f_{\bar z}\big].$$

And our assumption that $\operatorname{Re} f_{\bar z} \geqslant |f_z|$ implies that the last expression is nonnegative almost everywhere. We conclude that

$$\operatorname{Re}\big[(t - s_\varepsilon)(f(t) - f(s_\varepsilon))\big] \geqslant 0.$$

Case II. Either the line segment from s_ε to t is in T or it is not true that $\operatorname{Re} f_{\bar z} \geqslant |f_z|$ a.e. on that segment.

Let $X = \{w \text{ in } D: f_x(w) \text{ or } f_y(w)$ does not exist or, if they exist, $\operatorname{Re} f_{\bar z}(w) < |f_z(w)|\}$. For each $\alpha \in [0, 2\pi]$ denote by $L(\alpha)$ the radial (with center $= s$) segment joining $s + \varepsilon e^{i\alpha}$ to the unit circle and let $B = \{\alpha \in [0, 2\pi]: X \cap L(\alpha)$ has positive linear measure$\}$.

It is not hard to show that $m_1 B = 0$ where m_1 is Lebesgue measure on the real line; for instance, Fubini's theorem may be applied to this situation. Next, let $A = \{\alpha \in [0, 2\pi]: L(\alpha) \in T\}$. It will now be shown A has the property that if G is a measurable subset of A, then $m_1 G = 0$.

Let G be a measurable subset of A and let $Y = \{e^{i\alpha}: \alpha \in G\}$. Choose $R > \varepsilon$ so large that $D \subset D(s; R)$. Put $L = \{L(\alpha): \alpha \in G\}$ and L' = all extensions of segments in L to the circle $S(s; R)$. On the one hand, $M(L') = \sigma(Y)(\operatorname{Log} R/\varepsilon)^{-1}$ by Proposition 2.2 (and Theorem 2.1, part (i)); on the other hand $L \subset T$ and each path in L' has a subpath in L so that (again by Theorem 2.1) $M(L') \leqslant M(L) \leqslant M(T)$. Since T is from Theorem 2.3, $M(T) = 0$. Thus, $M(L') = 0$ and $\sigma(Y) = 0$; from which it follows that $m_1 G = 0$.

Now since any measurable subset of A has m_1-measure zero and since $m_1 B = 0$, any measurable subset of $A \cup B$ has m_1-measure zero. We conclude that $A \cup B$ contains no intervals. Therefore, the line segment from s_ε to t (although it may be in T) can be approximated arbitrarily close by radial segments which are not in T. Choose a_n, b_n on such line segments such that $a_n \in S(s; \varepsilon)$, $a_n \to s_\varepsilon$, $b_n \to t$. By Theorem 2.3, the Case I situation applies to yield $\operatorname{Re}[(b_n - a_n)(f(b_n) - f(a_n))] \geqslant 0$. The continuity of f implies $\operatorname{Re}[(t - s_\varepsilon)(f(t) - f(s_\varepsilon))] \geqslant 0$ and by the comments at the start of the proof, Theorem I is proved.

Next, we indicate a mild extension of Theorem I.

THEOREM. *Suppose g is a function which satisfies the hypothesis of Theorem* I *and suppose R is a function in $C(D)$ which satisfies $|R(t) - R(s)| \leqslant k|g(t) - g(s)|$ for all t, s in D with k a positive constant between* 0 *and* 1. *If $f = g + R$, then $P_f = C(D)$.*

An example of such a function R can be constructed by taking any real valued function of a real variable, say h, which satisfies $|h'| \leqslant k < 1$ and putting $R = h(\operatorname{Im} g) - ih(\operatorname{Re} g)$. In order to prove this extension, a couple of observations are relevant. First, for each $s \in D$, $f^{-1}(f(s)) = g^{-1}(g(s))$; this is a direct consequence of the Lipschitz condition on R. Since $g^{-1}(g(s))$ is countable, $f^{-1}(f(s))$ is also. Next, for each s the function $(z - s)(f - f(s))$ never takes a negative real number for a value in D. In fact, for any $\varepsilon > 0$ and any $t \in D$

$$
\begin{aligned}
&|(t - s)(f(t) - f(s)) + \varepsilon| \\
&\qquad = |(t - s)(g(t) - g(s)) + \varepsilon + (t - s)(R(t) - R(s))| \\
&\qquad \geqslant |(t - s)(g(t) - g(s)) + \varepsilon| - |(t - s)(R(t) - R(s))| \\
&\qquad \geqslant |(t - s)(g(t) - g(s)) + \varepsilon| - k|(t - s)(g(t) - g(s))|.
\end{aligned}
$$

And this last expression is positive since $\operatorname{Re}[(t - s)(g(t) - g(s))] \geqslant 0$. In order to finish a proof of the extension, it suffices to combine an argument used by J. Wermer in Lemma 3 of [7] with the argument in Theorem A of [4].

3. In order to have $P_f = C(D)$ it is not sufficient to assume that $f_{\bar z} \neq 0$ everywhere. This is demonstrated by an example of J. Wermer, namely, $f = \bar z \phi(|z|^2)$ where $\phi(t) = (t^2 - 1)/3 + i(t - 1)/2$. Straight calculation shows $f_{\bar z}$ vanishes nowhere in $\mathbf{C}$ and f itself vanishes on the unit circle. Hence, if $g \in P_f$, then g, restricted to the unit circle, is approximable by polynomials in z. It follows that $P_f \neq C(D)$.

This partially leads us to consider the class of functions which have the special form $f = \bar z^k \phi(|z|^{2k})$ where ϕ is a complex valued function of a real variable and where k is an arbitrary but fixed positive integer. For instance, some polynomials in z and $\bar z$ can be expressed in this form with ϕ being a polynomial in a real variable which has complex coefficients.

Before proving Theorem II, we supply additional information which will be needed.

THEOREM 3.1 (LAVRENTIEV). *Let X be a compact subset of* $\mathbf{C}$ *and let $P(X)$ denote the uniform closure on X of all polynomials in z. If X has empty interior* *and connected complement, then* $P(X) = C(X)$.

A functional-analytic proof of this result may be found in **[1]**.

THEOREM 3.2. *Let* X *be a compact subset of* $\mathbf{C}$ *and let* A *be a closed subalgebra of* $C(X)$. *Let* g *be a real valued function in* A *and for each real number* α *put* $X_\alpha = g^{-1}(\alpha)$. *If* A *restricted to each* X_α *is dense in* $C(X_\alpha)$ *for each* α, *then* $A = C(X)$.

This is a special case of a theorem due to Silov (and, in greater generality due to Bishop). For a simple proof which was discovered by Glicksberg and which is based on an argument of De Branges in **[2]**, the reader should see Theorem 2.7.5 in **[1]**.

LEMMA 3.3. *Let* f *be as in Theorem* II. *Then the function* $z^k f$ *maps* D *onto a Jordan arc* Γ *where distinct circles about* 0 *are mapped to distinct points of* Γ.

PROOF OF LEMMA 3.3. Straightforward calculation yields

$$f_z = k\bar{z}^{k+1}\phi'\left(|z|^{2k}\right)|z|^{2k-2}$$

and

$$f_{\bar{z}} = k\bar{z}^{k-1}\phi'\left(|z|^{2k}\right)|z|^{2k} + k\bar{z}^{k-1}\phi\left(|z|^{2k}\right)$$

in Int D. Hence,

$$|f_z|^2 - |f_{\bar{z}}|^2 = -k^2|z^{k-1}|^2\left(\left|\phi\left(|z|^{2k}\right)\right|^2 + 2\operatorname{Re}\left[|z|^{2k}\phi\left(|z|^{2k}\right)\overline{\phi'\left(|z|^{2k}\right)}\right]\right).$$

Let $t = |z|^{2k}$ and $H(t) = |\phi(t)|^2$. The hypotheses on f imply that $|\phi(t)|^2 + 2\operatorname{Re}[t\phi(t)\overline{\phi'(t)}] > 0$ or that $[tH(t)]' > 0$ when $0 < t < 1$. Thus, $t \to tH(t)$ is a strictly increasing function of t. Next, if $0 < r < s < 1$, then we have $r^2|\phi(r)|^2 < rsH(s) < s^2H(s) = s^2|\phi(s)|^2$; so that the map $t \to t|\phi(t)|$ is also a strictly increasing function. We conclude that the image of the closed unit interval under $t\phi$ is a Jordan arc Γ. Since $z^k f = |z|^{2k}\phi(|z|^{2k})$ the image of D under $z^k f$ is exactly Γ and distinct circles about 0 are mapped into distinct points of Γ.

PROOF OF THEOREM II. Let Γ be as in Lemma 3.3. Since Γ is a Jordan arc, it has connected complement and no interior. By Theorem 3.1, $P(\Gamma) = C(\Gamma)$. Let h be the inverse of the map $t \to t\phi(t)$ which traced out Γ and let $g = h \circ (z^k f)$. Since $h \in C(\Gamma)$, $h \in P(\Gamma)$ and so $g \in P_f$. Further, the level sets of g are precisely the circles in D with center at 0 and the restriction of P_f to any circle about the origin is dense in all continuous functions on that circle (P_f restricted to the circle contains z and $\bar{z}^k$, hence, z and $\bar{z}$). We conclude from Theorem 3.2 that $P_f = C(D)$.

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