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THE SIDON CONSTANT OF A FINITE ABELIAN GROUP

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ABSTRACT. It is shown that the Helson constant of a finite abelian group, G, is exactly $(\text{Card } G)^{1/2}$.

The purpose of this note is to prove the following theorem.

THEOREM. *Let G be a finite abelian group of cardinality n. Then there exists a nonzero measure μ on G such that* $\|\mu\|/\|\hat{\mu}\|_\infty = n^{1/2}$.

The *Sidon* (*or Helson*) constant $\alpha(E)$ of a finite set E of a locally compact abelian group is the supremum of the ratio $\|\mu\|/\|\hat{\mu}\|_\infty$ as μ ranges over nonzero measures concentrated on E. The Sidon constant for E is at most $(\text{Card } E)^{1/2}$ (see [**K**, p. 34]). Thus, the theorem establishes the claim of the abstract.

This result is a qualitative improvement of previous results. For example, [**LR**, pp. 78–80] shows that the Sidon constant of a finite abelian group is at least $(\text{Card } G/2e \log \text{Card } G)^{1/2}$. For a finite cyclic group, Shapiro-Rudin polynomials can be used to show that the Sidon constant of G is at least $2^{-3/2}(\text{Card } G)^{1/2}$ [**K**, p. 35]. Neither of these results can be improved by modification of the techniques used to obtain them.

PROOF OF THEOREM. It will be sufficient to prove the theorem in case that G is a finite cyclic group. Indeed, if the theorem holds for finite cyclic groups, and G is a finite product of cyclic groups, then the product of the measures "that work" for the factors of G has the required property.

We now exhibit the measure that has the required property in case that G is a finite cyclic group of order n. We identify G with the integers $1, 2, \ldots, n$ with addition modulo n.

If n is even, we let μ be the measure on G that has mass at j given by

$$\mu(j) = \exp(2\pi i j^2/2n), \quad \text{for } 1 \leqslant j \leqslant n. \tag{1}$$

Obviously $\|\mu\| = n$. We need to show that $\|\hat{\mu}\|_\infty = n^{1/2}$. Since $\|(\mu * \tilde{\mu})\hat{}\|_\infty = \|\hat{\mu}\|_\infty^2$, it will suffice to show that $\mu * \mu = n\delta$ where δ is the point mass at the identity. We calculate. The following formulae are easily established. (Recall that addition is mod n.)

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(2) $$\mu(-j) = \mu(n-j) = \mu(j).$$

(3) $$\tilde{\mu}(j) = \exp(-2\pi i\, j^2/2n).$$

We then have, for $1 \leqslant k \leqslant n$,

$$\begin{aligned}\mu * \tilde{\mu}(k) &= \sum_{j=1}^{n} \mu(k-j)\,\tilde{\mu}(j)\\ &= \sum_{j=1}^{n} \exp\Big(2\pi i\big[(k-j)^2 - j^2\big]/2n\Big)\\ &= \exp(2\pi i\, k^2/2n) \sum_{j=1}^{n} \exp(2\pi i(-kj/n)).\end{aligned}$$

Of course, when $1 \leqslant k < n$, the last sum is zero. Thus, $\mu * \mu = n\delta$.

For odd n, we use $\mu(j) = \exp(2\pi i\, j^2/n)$. Then $\tilde{\mu}(j) = \exp(-2\pi i\, j^2/n)$ and

$$\mu * \tilde{\mu}(k) = \exp(2\pi i\, k^2/n) \sum_{j=1}^{n} \exp(2\pi i\, 2kj/n).$$

Since, for odd n, $2k \equiv 0 \pmod n$ if and only if $k \equiv 0 \pmod n$, the last sum is zero when $1 \leqslant k < n$. Thus, $\mu * \tilde{\mu} = n\delta$.

REMARK. The corresponding problem for arithmetic progressions is much more difficult. It is not known if $\lim \alpha(\{1, 2, \ldots, n\})/n^{1/2} = 1$. See [N] for a discussion.

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