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PEAK SETS FOR LIPSCHITZ FUNCTIONS

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ABSTRACT. We study the peak sets for the algebras of functions analytic in the unit disc D and satisfying a Lipschitz condition on ∂D.

Let D denote the open unit disc in the complex plane, let $\bar{D}$ denote its closure, and let ∂D denote its boundary. For $0 < \alpha \leqslant 1$, let Lip α be the algebra of complex-valued functions f analytic on D, continuous on $\bar{D}$, and satisfying a Lipschitz condition of order α on ∂D:

$$|f(z) - f(w)| \leqslant K|z - w|^{\alpha} \qquad (z, w \in \partial D).$$

Say that a function f defined on $\bar{D}$ peaks on $E \subseteq \partial D$ if $f = 1$ on E and $|f| < 1$ on $\bar{D} \sim E$. Finally, say that $E \subseteq \partial D$ is a peak set for Lip α if some $f \in$ Lip α peaks on E. We are interested in characterizing the peak sets for Lip α. For $\alpha = 1$, the situation is easily described. It follows from our Theorem 1 that a peak set for Lip 1 must be finite. On the other hand, a result of B. A. Taylor and D. L. Williams [7] shows that any finite subset of ∂D is a peak set for Lip 1. (In fact, [7] shows that the peaking function may be chosen to be infinitely differentiable on ∂D.) For $0 < \alpha < 1$, though, the situation seems more difficult, and we do not have such a characterization. Theorems 2 and 3 below give, respectively, sufficient and necessary conditions that $E \subseteq \partial D$ be a peak set for Lip α ($0 < \alpha < 1$). These conditions lend support to our conjecture of a necessary and sufficient condition, which we give at the end of this paper.

Before beginning, we establish some notation. We shall be dealing with closed subsets E of ∂D, and we shall always assume, without loss of generality, that $-1 \in E$. For such an E, $\partial D \sim E$ is the union of a collection $\{(e^{ia_n}, e^{ib_n})\}$ of disjoint open arcs such that $-\pi \leqslant a_n < b_n \leqslant \pi$. We put $\varepsilon_n = b_n - a_n$ and, when E has been specified, shall use the a_n's, b_n's, and ε_n's without further comment. Now suppose that f is a continuous function defined on D. We put $\|f\|_\infty$ equal to the (possibly infinite) number $\sup\{|f(z)|: z \in D\}$ and, for $0 < r < 1$, write

$$M(r,f) = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} |f(re^{it})|\, dt.$$

THEOREM 1. *Suppose that g is analytic in D, continuous on $\bar{D}$, and that*

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Re $g > 0$ *in* D. *Let* N *be the number of zeroes that* g *has on* ∂D. *Then* $N - 1 \leqslant 2\pi \| g' \|_\infty^2 / |g(0)|^2$.

PROOF. Evidently we may assume that $\| g' \|_\infty < +\infty$ so that, in particular, $g' \in H^1$ and g is absolutely continuous on ∂D with $dg(e^{it})/dt = ie^{it} \lim_{r \to 1} g'(re^{it})$ for almost every $t \in [-\pi, \pi]$. (See Theorem 3.11 in [2].) Thus

$$(1) \qquad |g(e^{it}) - g(e^{is})| \leqslant \| g' \|_\infty |t - s|, \qquad -\pi \leqslant t, s \leqslant \pi.$$

Now let E be the zero set of g in ∂D. Given n, it follows from (1) and $g(e^{ia_n}) = g(e^{ib_n}) = 0$ that

(2) if $a_n < t < b_n$, then

$$|g(e^{it})| \leqslant \| g' \|_\infty \min\{t - a_n, b_n - t\} \leqslant \| g' \|_\infty \varepsilon_n / 2.$$

Let u and v be, respectively, the real and imaginary parts of g. Then the real part of $1/g$ is $u/|g|^2$. Thus

(3)

$$\frac{u(0)}{|g(0)|^2} = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} \frac{u(e^{it})}{|g(e^{it})|^2}\, dt = \left(\frac{1}{2\pi}\right) \sum_n \int_{a_n}^{b_n} \frac{u(e^{it})}{|g(e^{it})|^2}\, dt$$

$$\geqslant \left(\frac{1}{\pi \| g' \|_\infty^2}\right) \sum_n \left(\frac{1}{\varepsilon_n}\right) \left[\int_{a_n}^{(a_n + b_n)/2} \frac{u(e^{it})}{t - a_n}\, dt + \int_{(a_n + b_n)/2}^{b_n} \frac{u(e^{it})}{b_n - t}\, dt \right],$$

where the last inequality follows from (2).

It follows from (1) that the integral

$$\left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} u(e^{it}) \cot\left(\frac{a - t}{2}\right) dt$$

converges absolutely whenever $u(e^{ia}) = 0$. In fact, the formula for conjugate functions on the circle allows us to write, in this case,

$$v(e^{ia}) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} u(e^{it}) \cot((a - t)/2)\, dt + v(0).$$

Since, in particular,

$$u(e^{ia_n}) = u(e^{ib_n}) = v(e^{ia_n}) = v(e^{ib_n}) = 0,$$

we have

$$\int_{-\pi}^{\pi} u(e^{it}) \left[\cot\left(\frac{a_n - t}{2}\right) - \cot\left(\frac{b_n - t}{2}\right) \right] dt = 0.$$

Rewriting this we get

$$\int_{-\pi}^{a_n} + \int_{b_n}^{\pi} u(e^{it}) \left[\cot\left(\frac{a_n - t}{2}\right) - \cot\left(\frac{b_n - t}{2}\right) \right] dt$$

$$(4) \qquad = \int_{a_n}^{b_n} u(e^{it}) \left[\cot\left(\frac{t - a_n}{2}\right) + \cot\left(\frac{b_n - t}{2}\right) \right] dt.$$

To examine the right-hand side of (4), we first note that for $a_n \leqslant t \leqslant (a_n +$

$b_n)/2$ we have

$$\left|\cot\left(\frac{b_n - t}{2}\right)\right| \leqslant \cot\left(\frac{t - a_n}{2}\right)$$
$$\leqslant \frac{\pi}{(t - a_n)}.$$

Thus

$$\begin{aligned}\text{(5)}\quad &\int_{a_n}^{(a_n+b_n)/2} u(e^{it})\left[\cot\left(\frac{t-a_n}{2}\right) + \cot\left(\frac{b_n - t}{2}\right)\right] dt \\ &\leqslant 2\pi \int_{a_n}^{(a_n+b_n)/2} \frac{u(e^{it})}{t - a_n}\, dt.\end{aligned}$$

Similarly,

$$\begin{aligned}\text{(6)}\quad &\int_{(a_n+b_n)/2}^{b_n} u(e^{it})\left[\cot\left(\frac{t-a_n}{2}\right) + \cot\left(\frac{b_n - t}{2}\right)\right] dt \\ &\leqslant 2\pi \int_{(a_n+b_n)/2}^{b_n} \frac{u(e^{it})}{b_n - t}\, dt.\end{aligned}$$

Now (3), (5), and (6) yield

$$\frac{u(0)}{|g(0)|^2} \geqslant \frac{1}{2\pi^2\|g'\|_\infty^2} \sum_n \frac{1}{\varepsilon_n} \int_{a_n}^{b_n} u(e^{it})\left[\cot\left(\frac{t-a_n}{2}\right) + \cot\left(\frac{b_n - t}{2}\right)\right] dt.$$

Taking into account (4), we have

(7)

$$\begin{aligned} &u(0)/|g(0)|^2 \\ &\geqslant \frac{1}{2\pi^2\|g'\|_\infty^2} \sum_n \frac{1}{\varepsilon_n} \int_{-\pi}^{a_n} + \int_{b_n}^{\pi} u(e^{it})\left[\cot\left(\frac{a_n - t}{2}\right) - \cot\left(\frac{b_n - t}{2}\right)\right] dt \\ &= \frac{1}{2\pi^2\|g'\|_\infty^2} \int_{-\pi}^{\pi} u(e^{it})\left\{\sum_n \frac{1}{\varepsilon_n}\left[\cot\left(\frac{a_n - t}{2}\right) - \cot\left(\frac{b_n - t}{2}\right)\right]\right. \\ &\qquad\qquad \left.\cdot\, \chi_{[-\pi, a_n]\cup[b_n,\pi]}(t)\right\} dt,\end{aligned}$$

where $\chi_{[-\pi,a_n]\cup[b_n,\pi]}$ is the characteristic function of the set $[-\pi, a_n] \cup [b_n, \pi]$. Now if $-\pi \leqslant t \leqslant a_n$ or $b_n \leqslant t \leqslant \pi$, it is easy to check that

$$\cot\left(\frac{a_n - t}{2}\right) - \cot\left(\frac{b_n - t}{2}\right) \geqslant \frac{\varepsilon_n}{2}.$$

Thus if $t \in (a_m, b_m)$, the quantity $\{\cdots\}$ in the last term of (7) is not less than $\sum_{n\neq m} 1/2$. Letting N be the cardinality (*a priori* possibly $+\infty$) of the collection $\{(e^{ia_n}, e^{ib_n})\}$, and noting that almost every $t \in [-\pi, \pi]$ is in some (a_m, b_m), we see from (7) that

$$\frac{u(0)}{|g(0)|^2} \geqslant \frac{N-1}{2\pi\|g'\|_\infty^2}\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} u(e^{it})\,dt = \frac{(N-1)u(0)}{2\pi\|g'\|_\infty^2}.$$

This finishes the proof of the theorem.

COROLLARY. *If E is a peak set for* Lip 1, *then E is finite.*

PROOF. Suppose that $f \in$ Lip 1 peaks on E. It follows from a result of Hardy and Littlewood (see Theorem 5.1 in [2]) that $\|f'\|_\infty < \infty$, so Theorem 1 applies to $g = 1 - f$.

Theorem 1 is a quantitative version of the following statement: if g is analytic in D, continuous on $\overline{D}$, and has positive real part on D, and if $\|g'\|_\infty$ is finite, then the zero set of g is finite. If the hypotheses on g are strengthened to require that g' be continuous on $\overline{D}$, this statement is proved in [1]. It is perhaps surprising how much more difficult the proof becomes when f' is not required to be continuous on $\overline{D}$.

The proof of our next theorem is similar to the proof of Theorem 5 in [1].

THEOREM 2. *Suppose $0 < \alpha < 1$ and that $E \subseteq \partial D$ is a closed set of measure zero satisfying $\sum_n \varepsilon_n^{(1-\alpha)/(3-\alpha)} < +\infty$. Then E is a peak set for* Lip α.

PROOF. Put $\gamma = 2/(3 - \alpha)$ and define ϕ on ∂D by

$$\phi(e^{it}) = \begin{cases} (t - a_n)^{-\gamma} + (b_n - t)^{-\gamma} & \text{if } a_n < t < b_n, \\ +\infty & \text{if } e^{it} \in E. \end{cases}$$

Our hypothesis $\sum_n \varepsilon_n^{(1-\alpha)/(3-\alpha)} < +\infty$ implies that the function $t \mapsto \phi(e^{it})$ is integrable on $[-\pi, \pi]$. Next define an analytic function g on D by

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{it})\,\frac{e^{it} + z}{e^{it} - z}\,dt.$$

The properties of g that we need are these:

(i) g has a continuous extension to $\overline{D} \sim E$,

(ii) Re $g(re^{it}) \to \phi(e^{it})$ as $r \to 1$,

(iii) $|g'(z)| \leqslant M[\text{dist}(z, E)]^{-2}$ for some constant M,

(iv) $g' \exp(-g)$ is bounded on D.

For (i) and (ii) see [5, p. 80]; for (iii) see [8, Lemma 2.3]; and for (iv) see [6, pp. 1270–1271]. Now put $h = 1/g$ and $f = \exp(-h)$. Then h has a continuous extension to $\overline{D}$, h has positive real part on $\overline{D} \sim E$ (because g does), and $h(e^{it}) = 0$ for $e^{it} \in E$. Thus f peaks on E and all that remains is to show that $f \in$ Lip α. This will be done by showing that h' (and thus f') belongs to the Hardy class H^p, where $p = 1/(1 - \alpha)$. (See, for example, exercise 9 on p. 91 in [2].)

Now $h' = -h^2 g' \exp(-g)/\exp(-g)$, so we see that h' is the quotient of two bounded analytic functions. Thus h' belongs to the Nevanlinna class N. But because $\exp(-g)$ is an outer function, $h' \in N^+$. Thus $h' \in H^p$ provided the boundary function $h'(e^{it}) \in L^p(\partial D)$. (See [2, Theorem 2.11].) Now

$$|h'(re^{it})| = |g'(re^{it})| \cdot |g(re^{it})|^{-2}$$
$$\leqslant M\big[\operatorname{dist}(re^{it}, E)\big]^{-2}\big[\operatorname{Re} g(re^{it})\big]^{-2}.$$

Thus it follows from (ii) and (iii) above that

$$\limsup_{r\to 1} |h'(re^{it})| \leqslant M\big[\operatorname{dist}(e^{it}, E)\big]^{-2}\big[\phi(e^{it})\big]^{-2}. \tag{8}$$

There is a constant K such that $[\operatorname{dist}(e^{it}, E)]^{-\gamma} \leqslant K\phi(e^{it})$, and so

$$\big[\operatorname{dist}(e^{it}, E)\big]^{-2} \leqslant \big[K\phi(e^{it})\big]^{2/\gamma}.$$

When combined with (8), this implies that

$$|h'(e^{it})| \leqslant MK^{2/\gamma}\big[\phi(e^{it})\big]^{(2/\gamma)-2} \quad \text{a.e.}$$

Consequently,

$$|h'(e^{it})|^{p} \leqslant \text{constant} \cdot \phi(e^{it}) \quad \text{a.e.}$$

This shows that $h'(e^{it}) \in L^{p}(\partial D)$ and so completes the proof of the theorem.

To prove our final theorem we require a lemma.

LEMMA. *Let g be analytic on D and have positive real part. Then*

$$M(r, g) = O\big(\log[1/(1-r)]\big) \quad \textit{as } r \to 1.$$

PROOF. This is a consequence of [**3**, Theorem 7].

THEOREM 3. *Fix α with $0 < \alpha < 1$ and suppose $E \subseteq \partial D$ is a peak set for* Lip α. *Then for each $\delta > 1$ we have*

$$\sum_n \varepsilon_n^{1-\alpha}|\log(1/\varepsilon_n)|^{-\delta} < +\infty.$$

PROOF. Let $\delta > 1$ be given. Let us assume, without loss of generality, that $b_n - a_n < \pi/2$ for each n so that if r_0 is the smallest of the numbers $\cos(b_n - a_n)$, then $0 < r_0 < 1$. Since E is a peak set for Lip α, there exists $f \in$ Lip α having positive real part on $\partial D \sim E$ with $f(e^{it}) = 0$ for $e^{it} \in E$. Let K be a Lipschitz constant for f on $\bar{D}$ so that $|f(z) - f(w)| \leqslant K|z - w|^{\alpha}$ ($z, w \in \bar{D}$). (The assumption that $f \in$ Lip α means that f satisfies a Lipschitz condition on ∂D. But an old result of Hardy and Littlewood [**4**, Theorem 41] shows that f is then Lipschitz on $\bar{D}$.)

By elementary calculus, $\int_0^1 1/(1-r)[\log 1/(1-r)]^{\delta}\, dr < +\infty$. When combined with the Lemma as applied to $g = 1/f$, this yields

$$\int_{r_0}^{1} M(r, 1/f)/(1-r)\big[\log 1/(1-r)\big]^{1+\delta}\, dr < +\infty.$$

Thus,

$$+\infty > \int_{r_0}^{1}\left[\int_{-\pi}^{\pi} \frac{1/|f(re^{it})|}{(1-r)[\log 1/(1-r)]^{1+\delta}}\,dt\right]dr$$

$$= \int_{r_0}^{1}\left[\sum_n \int_{a_n}^{b_n} \frac{1/|f(re^{it})-f(e^{ia_n})|}{(1-r)[\log 1/(1-r)]^{1+\delta}}\,dt\right]dr$$

$$= \sum_n \int_{a_n}^{b_n}\left[\int_{r_0}^{1} \frac{dr}{|f(re^{it})-f(e^{ia_n})|(1-r)[\log 1/(1-r)]^{1+\delta}}\right]dt$$

$$\geqslant \sum_n \int_{a_n}^{b_n}\left[\int_{r_0}^{1} \frac{dr}{K|re^{it}-e^{ia_n}|^{\alpha}(1-r)[\log 1/(1-r)]^{1+\delta}}\right]dt$$

$$\geqslant \sum_n \int_{a_n}^{b_n}\left[\int_{\cos(t-a_n)}^{1} \frac{dr}{K|e^{it}-e^{ia_n}|^{\alpha}(1-r)[\log 1/(1-r)]^{1+\delta}}\right]dt.$$

(The last inequality follows from the facts $r_0 \leqslant \cos(b_n - a_n) \leqslant \cos(t - a_n)$ if $a_n \leqslant t \leqslant b_n$, and $|re^{it} - e^{ia_n}| \leqslant |e^{it} - e^{ia_n}|$ if $\cos(t - a_n) \leqslant r \leqslant 1$). Evaluating $\int_{\cos(t-a_n)}^{1} 1/(1-r)[\log 1/(1-r)]^{1+\delta}\,dr$, we see that the last sum above is equal to

$$(1/K\delta)\sum_n \int_{a_n}^{b_n} \frac{dt}{|e^{it}-e^{ia_n}|^{\alpha}[\log 1/(1-\cos(t-a_n))]^{\delta}}$$

$$\geqslant \left(\frac{1}{2^{\delta}K\delta}\right)\sum_n \int_{a_n}^{b_n} \frac{dt}{(t-a_n)^{\alpha}[\log \pi/\sqrt{2}\,(t-a_n)]^{\delta}},$$

where in obtaining the inequality we have used the relations

$$|e^{it} - e^{ia_n}| \leqslant t - a_n \quad \text{and} \quad 1 - \cos(t - a_n) \geqslant 2(t - a_n)^2/\pi^2.$$

It follows, after a change of variable, that

$$+\infty > \sum_n \int_0^{\sqrt{2}\,\varepsilon_n/\pi} t^{-\alpha}[\log 1/t]^{-\delta}\,dt. \tag{9}$$

Consider the equation

$$(d/dt)\left(t^{1-\alpha}[\log 1/t]^{-\delta}\right)$$
$$= (1-\alpha)t^{-\alpha}(\log 1/t)^{-\delta} + \delta t^{-\alpha}(\log 1/t)^{-\delta-1}.$$

By integrating both sides of this equation from 0 to $\sqrt{2}\,\varepsilon_n/\pi$ and then summing over n, we obtain

$$\left(\frac{\sqrt{2}}{\pi}\right)^{1-\alpha}\sum_n \varepsilon_n^{1-\alpha}\left(\log\frac{\pi}{\sqrt{2}\,\varepsilon_n}\right)^{-\delta}$$
$$= (1-\alpha)\sum_n \int_0^{\sqrt{2}\,\varepsilon_n/\pi} t^{-\alpha}\left(\log\frac{1}{t}\right)^{-\delta} dt$$
$$+\delta\sum_n \int_0^{\sqrt{2}\,\varepsilon_n/\pi} t^{-\alpha}\left(\log\frac{1}{t}\right)^{-\delta-1} dt.$$

Each sum on the right-hand side of this equation is finite because of (9). Thus so is the left-hand side. It follows easily that $\sum_n \varepsilon_n |\log 1/\varepsilon_n|^{-\delta} + \infty$. This completes the proof of the theorem.

In conclusion, we conjecture that the condition $\sum_n \varepsilon_n^{1-\alpha} < +\infty$ is necessary and sufficient for a closed subset E of ∂D having measure 0 to be a peak set for Lip α.

References

1. D. J. Caveny and W. P. Novinger, *Boundary zeros of functions with derivative in H^p*, Proc. Amer. Math. Soc. **25** (1970), 776–780.
2. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
3. G. H. Hardy and J. E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math. **167** (1931), 405–423.
4. ______, *Some properties of fractional integrals*. II, Math. Z. **34** (1932), 403–439.
5. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
6. B. A. Taylor and D. L. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Canad. J. Math. **22** (1970), 1266–1283.
7. ______, *The peak sets of A^m*, Proc. Amer. Math. Soc. **24** (1970), 604–606.
8. ______, *Zeros of Lipschitz functions analytic in the unit disc*, Michigan Math. J. **18** (1971), 129–139.

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