PEAK SETS FOR LIPSCHITZ FUNCTIONS

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Abstract. We study the peak sets for the algebras of functions analytic in the unit disc D and satisfying a Lipschitz condition on ∂D.

Let D denote the open unit disc in the complex plane, let D denote its closure, and let ∂D denote its boundary. For 0 < α < 1, let Lip α be the algebra of complex-valued functions f analytic on D, continuous on D, and satisfying a Lipschitz condition of order α on ∂D:

|f(z) - f(w)| < K|z - w|^{α} \quad (z, w \in \partial D).

Say that a function f defined on D peaks on E ⊆ ∂D if f = 1 on E and |f| < 1 on D \setminus E. Finally, say that E ⊆ ∂D is a peak set for Lip α if some f ∈ Lip α peaks on E. We are interested in characterizing the peak sets for Lip α. For α = 1, the situation is easily described. It follows from our Theorem 1 that a peak set for Lip 1 must be finite. On the other hand, a result of B. A. Taylor and D. L. Williams [7] shows that any finite subset of ∂D is a peak set for Lip 1. (In fact, [7] shows that the peaking function may be chosen to be infinitely differentiable on ∂D.) For 0 < α < 1, though, the situation seems more difficult, and we do not have such a characterization. Theorems 2 and 3 below give, respectively, sufficient and necessary conditions that E ⊆ ∂D be a peak set for Lip α (0 < α < 1). These conditions lend support to our conjecture of a necessary and sufficient condition, which we give at the end of this paper.

Before beginning, we establish some notation. We shall be dealing with closed subsets E of ∂D, and we shall always assume, without loss of generality, that -1 ∈ E. For such an E, ∂D \sim E is the union of a collection \{(a_n, b_n)\} of disjoint open arcs such that -π < a_n < b_n < π. We put \epsilon_n = b_n - a_n and, when E has been specified, shall use the a_n’s, b_n’s, and \epsilon_n’s without further comment. Now suppose that f is a continuous function defined on D. We put \|f\|_\infty equal to the (possibly infinite) number sup{|f(z)|: z ∈ D} and, for 0 < r < 1, write

M(r, f) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} |f(re^{it})| \, dt.

Theorem 1. Suppose that g is analytic in D, continuous on D, and that
Re \( g > 0 \) in \( D \). Let \( N \) be the number of zeroes that \( g \) has on \( \partial D \). Then
\[ N - 1 < 2\pi \| g' \|_{\infty}^2 / |g(0)|^2. \]

**Proof.** Evidently we may assume that \( \| g' \|_{\infty} < +\infty \) so that, in particular, 
\( g' \in H^1 \) and \( g \) is absolutely continuous on \( \partial D \) with 
\[ dg(e^{it})/dt = ie^{it}\lim_{r \to 1} g(re^{it}) \]
for almost every \( t \in [-\pi, \pi] \). (See Theorem 3.11 in [2].) Thus
\[
|g(e^{it}) - g(e^{is})| < \| g' \|_{\infty}|t - s|, \quad -\pi < t, s < \pi.
\]
Now let \( E \) be the zero set of \( g \) in \( \partial D \). Given \( n \), it follows from (1) and 
\( g(e^{ia_n}) = g(e^{ib_n}) = 0 \) that
\[
(2) \text{ if } a_n < t < b_n, \text{ then}
\]
\[
|g(e^{it})| < \| g' \|_{\infty}\min \{ t - a_n, b_n - t \} < \| g' \|_{\infty}\varepsilon_n / 2.
\]
Let \( u \) and \( v \) be, respectively, the real and imaginary parts of \( g \). Then the
real part of \( 1/g \) is \( u/|g|^2 \). Thus
\[
(3) \quad \frac{u(0)}{|g(0)|^2} = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} \frac{u(e^{it})}{|g(e^{it})|^2} \, dt = \left( \frac{1}{2\pi} \right) \sum_n \int_{a_n}^{b_n} \frac{u(e^{it})}{|g(e^{it})|^2} \, dt
\]
\[
> \left( \frac{1}{\pi} \right) \left( \frac{1}{\| g' \|_{\infty}} \right) \sum_n \left( \frac{1}{\varepsilon_n} \right) \left[ \int_{a_n}^{(a_n + b_n)/2} \frac{u(e^{it})}{t - a_n} \, dt + \int_{(a_n + b_n)/2}^{b_n} \frac{u(e^{it})}{b_n - t} \, dt \right],
\]
where the last inequality follows from (2).
It follows from (1) that the integral
\[
\left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} u(e^{it})\cot\left( \frac{a - t}{2} \right) \, dt
\]
converges absolutely whenever \( u(e^{ia_n}) = 0 \). In fact, the formula for conjugate 
functions on the circle allows us to write, in this case,
\[
v(e^{ia}) = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} u(e^{it})\cot((a - t)/2) \, dt + v(0).
\]
Since, in particular, 
\[
u(e^{ia_n}) = u(e^{ib_n}) = v(e^{ia_n}) = v(e^{ib_n}) = 0,
\]
we have
\[
\int_{-\pi}^{\pi} u(e^{it}) \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] \, dt = 0.
\]
Rewriting this we get
\[
\int_{-\pi}^{a_n} + \int_{a_n}^{b_n} u(e^{it}) \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] \, dt
\]
\[
= \int_{a_n}^{b_n} u(e^{it}) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] \, dt.
\]
To examine the right-hand side of (4), we first note that for \( a_n < t < (a_n +
\[ \frac{b_n}{2} \text{ we have} \]
\[ \left| \cot \left( \frac{b_n - t}{2} \right) \right| < \cot \left( \frac{t - a_n}{2} \right) < \frac{\pi}{(t - a_n)}. \]

Thus
\[ \int_{a_n}^{(a_n + b_n)/2} u(e^t) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt \]
\[ < 2\pi \int_{a_n}^{(a_n + b_n)/2} \frac{u(e^t)}{t - a_n} dt. \]

Similarly,
\[ \int_{(a_n + b_n)/2}^{b_n} u(e^t) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt \]
\[ < 2\pi \int_{(a_n + b_n)/2}^{b_n} \frac{u(e^t)}{b_n - t} dt. \]

Now (3), (5), and (6) yield
\[ \frac{u(0)}{|g(0)|^2} > \frac{1}{2\pi^2 \| g' \|_\infty^2} \sum_n \frac{1}{\epsilon_n} \int_{a_n}^{b_n} u(e^t) \left[ \cot \left( \frac{t - a_n}{2} \right) + \cot \left( \frac{b_n - t}{2} \right) \right] dt. \]

Taking into account (4), we have
\[ u(0)/|g(0)|^2 > \frac{1}{2\pi^2 \| g' \|_\infty^2} \sum_n \frac{1}{\epsilon_n} \int_{-\pi}^\pi u(e^t) \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] dt \]
\[ = \frac{1}{2\pi^2 \| g' \|_\infty^2} \int_{-\pi}^\pi u(e^t) \left( \sum_n \frac{1}{\epsilon_n} \left[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) \right] \right) \chi_t(t) \]
\[ \cdot \chi_t(-\pi, a_n) \cup [b_n, \pi](t) \]

where \(\chi_t(-\pi, a_n) \cup [b_n, \pi]\) is the characteristic function of the set \([-\pi, a_n] \cup [b_n, \pi]\].

Now if \(-\pi < t < a_n\) or \(b_n < t < \pi\), it is easy to check that
\[ \cot \left( \frac{a_n - t}{2} \right) - \cot \left( \frac{b_n - t}{2} \right) > \epsilon_n. \]

Thus if \(t \in (a_m, b_m)\), the quantity \(\cdots\) in the last term of (7) is not less than \(\sum_{n \neq m} 1/2\). Letting \(N\) be the cardinality (a priori possibly +\(\infty\)) of the collection \(\{(e^{i\alpha_n}, e^{i\beta_n})\}\) and noting that almost every \(t \in [-\pi, \pi]\) is in some \((a_m, b_m)\), we see from (7) that
This finishes the proof of the theorem.

**Corollary.** If \( E \) is a peak set for \( \text{Lip} 1 \), then \( E \) is finite.

**Proof.** Suppose that \( f \in \text{Lip} 1 \) peaks on \( E \). It follows from a result of Hardy and Littlewood (see Theorem 5.1 in [2]) that \( \|f'\|_\infty < \infty \), so Theorem 1 applies to \( g = 1 - f \).

Theorem 1 is a quantitative version of the following statement: if \( g \) is analytic in \( D \), continuous on \( D \), and has positive real part on \( D \), and if \( \|g'\|_\infty \) is finite, then the zero set of \( g \) is finite. If the hypotheses on \( g \) are strengthened to require that \( g' \) be continuous on \( D \), this statement is proved in [1]. It is perhaps surprising how much more difficult the proof becomes when \( f' \) is not required to be continuous on \( D \).

The proof of our next theorem is similar to the proof of Theorem 5 in [1].

**Theorem 2.** Suppose \( 0 < \alpha < 1 \) and that \( E \subseteq \partial D \) is a closed set of measure zero satisfying \( \sum_n e_n^{(1-\alpha)/(3-\alpha)} < +\infty \). Then \( E \) is a peak set for \( \text{Lip} \alpha \).

**Proof.** Put \( \gamma = 2/(3 - \alpha) \) and define \( \phi \) on \( \partial D \) by

\[
\phi(e^{it}) = \begin{cases} 
(t - a_n)^{-\gamma} + (b_n - t)^{-\gamma} & \text{if } a_n < t < b_n, \\
+\infty & \text{if } e^{it} \in E.
\end{cases}
\]

Our hypothesis \( \sum_n e_n^{(1-\alpha)/(3-\alpha)} < +\infty \) implies that the function \( t \mapsto \phi(e^{it}) \) is integrable on \( [-\pi, \pi] \). Next define an analytic function \( g \) on \( D \) by

\[
g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{it}) \frac{e^{it} + z}{e^{it} - z} \, dt.
\]

The properties of \( g \) that we need are these:

(i) \( g \) has a continuous extension to \( \overline{D} \sim E \),

(ii) \( \text{Re } g(re^{it}) \to \phi(e^{it}) \) as \( r \to 1 \),

(iii) \( |g'(z)| \leq M[\text{dist}(z, E)]^{-2} \) for some constant \( M \),

(iv) \( g' \exp(-g) \) is bounded on \( D \).

For (i) and (ii) see [5, p. 80]; for (iii) see [8, Lemma 2.3]; and for (iv) see [6, pp. 1270–1271]. Now put \( h = 1/g \) and \( f = \exp(-h) \). Then \( h \) has a continuous extension to \( \overline{D} \), \( h \) has positive real part on \( \overline{D} \sim E \) (because \( g \) does), and \( h(e^{it}) = 0 \) for \( e^{it} \in E \). Thus \( f \) peaks on \( E \) and all that remains is to show that \( f \in \text{Lip} \alpha \). This will be done by showing that \( h' \) (and thus \( f' \)) belongs to the Hardy class \( H^p \), where \( p = 1/(1 - \alpha) \). (See, for example, exercise 9 on p. 91 in [2].)

Now \( h' = -h^2g'\exp(-g)/\exp(-g) \), so we see that \( h' \) is the quotient of two bounded analytic functions. Thus \( h' \) belongs to the Nevanlinna class \( N \). But because \( \exp(-g) \) is an outer function, \( h' \in N^{+} \). Thus \( h' \in H^p \) provided the boundary function \( h'(e^{it}) \in L^p(\partial D) \). (See [2, Theorem 2.11].) Now
Thus it follows from (ii) and (iii) above that

$$\limsup_{r \to 1} |h'(re^\theta)| < M[D\text{dist}(e^\theta, E)]^{-2}[\phi(e^\theta)]^{-2}.$$

There is a constant $K$ such that $[\text{dist}(e^\theta, E)]^{-\gamma} < K\phi(e^\theta)$, and so

$$[\text{dist}(e^\theta, E)]^{-2} < K\phi(e^\theta)^{2/\gamma}.$$

When combined with (8), this implies that

$$|h'(e^\theta)| < MK^{2/\gamma}[\phi(e^\theta)]^{(2/\gamma) - 2} \quad \text{a.e.}$$

Consequently,

$$|h'(e^\theta)|^\theta < \text{constant} \cdot \phi(e^\theta) \quad \text{a.e.}$$

This shows that $h'(e^\theta) \in L^\theta(\partial D)$ and so completes the proof of the theorem.

To prove our final theorem we require a lemma.

**Lemma.** Let $g$ be analytic on $D$ and have positive real part. Then

$$M(r, g) = O\left(\log\left[\frac{1}{1 - r}\right]\right) \quad \text{as} \quad r \to 1.$$ 

**Proof.** This is a consequence of [3, Theorem 7].

**Theorem 3.** Fix $\alpha$ with $0 < \alpha < 1$ and suppose $E \subseteq \partial D$ is a peak set for Lip $\alpha$. Then for each $\delta > 1$ we have

$$\sum_n e_n^{-\alpha}(|\log(1/e_n)|^{-\delta} < +\infty.$$ 

**Proof.** Let $\delta > 1$ be given. Let us assume, without loss of generality, that $b_n - a_n < \pi/2$ for each $n$ so that if $r_0$ is the smallest of the numbers $\cos(b_n - a_n)$, then $0 < r_0 < 1$. Since $E$ is a peak set for Lip $\alpha$, there exists $f \in \text{Lip } \alpha$ having positive real part on $\partial D \sim E$ with $f(e^\theta) = 0$ for $e^\theta \in E$. Let $K$ be a Lipschitz constant for $f$ on $\overline{D}$ so that $|f(z) - f(w)| < K|z - w|^\alpha (z, w \in \overline{D})$. (The assumption that $f \in \text{Lip } \alpha$ means that $f$ satisfies a Lipschitz condition on $\partial D$. But an old result of Hardy and Littlewood [4, Theorem 41] shows that $f$ is then Lipschitz on $\overline{D}$.)

By elementary calculus, $\int_0^1/(1 - r)[\log 1/(1 - r)]^\delta dr < +\infty$. When combined with the Lemma as applied to $g = 1/f$, this yields

$$\int_{r_0}^1 M(r, 1/f)/(1 - r)[\log 1/(1 - r)]^\delta dr < +\infty.$$ 

Thus,
\[
+ \infty > \int_{r_0}^{+\infty} \left\{ \int_{-\pi}^{\pi} \frac{1/|f(re^{it})|}{(1 - r)[\log 1/(1 - r)]^{1+\delta}} \, dt \right\} \, dr
\]

\[
= \int_{r_0}^{+\infty} \left( \sum_n \int_{a_n}^{b_n} \frac{1/|f(re^{it}) - f(e^{ia_n})|}{(1 - r)[\log 1/(1 - r)]^{1+\delta}} \, dt \right) \, dr
\]

\[
= \sum_n \int_{a_n}^{b_n} \left( \int_{r_0}^{1} \frac{dr}{K|e^{re^{it}} - e^{ia_n}|(1 - r)[\log 1/(1 - r)]^{1+\delta}} \right) \, dt
\]

\[
> \sum_n \int_{a_n}^{b_n} \left( \int_{r_0}^{1} \frac{dr}{K|e^{re^{it}} - e^{ia_n}|(1 - r)[\log 1/(1 - r)]^{1+\delta}} \right) \, dt
\]

(The last inequality follows from the facts \(r_0 < \cos(b_n - a_n) < \cos(t - a_n)\) if \(a_n < t < b_n\), and \(|re^{it} - e^{ia_n}| < |e^{it} - e^{ia_n}|\) if \(\cos(t - a_n) < r < 1\). Evaluating \(\int_{\cos(t-a_n)}^{1}/(1 - r)[\log 1/(1 - r)]^{1+\delta} \, dr\), we see that the last sum above is equal to

\[
(1/K\delta) \sum_n \int_{a_n}^{b_n} \frac{dt}{|e^{it} - e^{ia_n}|[\log 1/(1 - \cos(t - a_n))]^{\delta}}
\]

\[
> \left( \frac{1}{2^\delta K\delta} \right) \sum_n \int_{a_n}^{b_n} \frac{dt}{(t - a_n)^\delta[\log \pi/\sqrt{2}(t - a_n)]^{\delta}}
\]

where in obtaining the inequality we have used the relations

\[|e^{it} - e^{ia_n}| < t - a_n \text{ and } 1 - \cos(t - a_n) > 2(t - a_n)^2/\pi^2.\]

It follows, after a change of variable, that

\[
+ \infty > \sum_n \int_0^{\sqrt{2} \epsilon_n/\pi} t^{-\alpha}[\log 1/t]^{-\delta} \, dt
\]

Consider the equation

\[
(d/dt)(t^{1-\alpha}[\log 1/t]^{-\delta}) = (1 - \alpha)t^{-\alpha}(\log 1/t)^{-\delta} + \delta t^{-\alpha}(\log 1/t)^{-\delta-1}.
\]

By integrating both sides of this equation from 0 to \(\sqrt{2} \epsilon_n/\pi\) and then summing over \(n\), we obtain
Each sum on the right-hand side of this equation is finite because of (9). Thus so is the left-hand side. It follows easily that \( \sum_n e_n^{1-\alpha} \left( \log \frac{\pi}{\sqrt{2} e_n} \right)^{-\delta} \) completes the proof of the theorem.

In conclusion, we conjecture that the condition \( \sum_n e_n^{1-\alpha} < +\infty \) is necessary and sufficient for a closed subset \( E \) of \( \partial D \) having measure 0 to be a peak set for Lip \( \alpha \).

REFERENCES


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