PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 1, January 1978

MAXIMAL AND MINIMAL RING TOPOLOGIES

NIEL SHELL[1]

ABSTRACT. An explicit description is given of a nondiscrete ring topology on the field Q of rational numbers which is strictly finer than the locally bounded topology on Q having the ring of integers as a preorder. It is observed that either there exist nonvaluable minimal ring topologies or there exist ring topologies containing no minimal ring topologies.

It is shown by example that the strongest nondiscrete non-Archimedean ring topology on the field of rational numbers, which is also maximal among all nondiscrete locally bounded ring topologies on the rationals, is not maximal among *all* nondiscrete ring topologies on the rationals. The existence of minimal and maximal ring topologies is discussed briefly.

The term *maximal* (*minimal*) topology on a ring will mean a topology that is maximal (minimal) in the lattice under set inclusion of all nontrivial, nondiscrete ring topologies on the ring. A ring topology is called *non-Archimedean* if its uniformity is non-Archimedean (see **[5]**), i.e., if there exists a base at zero of open additive subgroups. If A and B are subsets of a ring, $A - B = \{a - b | a \in A, b \in B\}$.

THEOREM 1. *Every nontrivial, nondiscrete ring topology is contained in a maximal topology.*

PROOF. If T is a totally ordered collection of nondiscrete ring topologies stronger than a given ring topology, then $\mathfrak{N} = \bigcup_{\mathfrak{T} \in T} \mathfrak{N}_{\mathfrak{T}}$, where $\mathfrak{N}_{\mathfrak{T}}$ is the $\mathfrak{T}$-neighborhood filter at zero, is the neighborhood filter at zero of a ring topology stronger than any topology in T. The topology determined by $\mathfrak{N}$ is nondiscrete because $\{0\} \notin \mathfrak{N}_{\mathfrak{T}}$ for any $\mathfrak{T} \in T$. The theorem now follows by Zorn's lemma.

COROLLARY 1.1. *There are maximal topologies on every infinite field.*

PROOF. Theorem 5.2 of **[3]** states that there are nontrivial, nondiscrete ring topologies on every infinite field.

The locally bounded topology $\mathfrak{T}_Z$ which has the integers Z as a preorder is

Received by the editors May 20, 1977.

AMS (MOS) subject classifications (1970). Primary 12J99; Secondary 06A20, 13J99, 54A10, 54E15.

Key words and phrases. Field topology, non-Archimedean ring topology, non-Archimedean uniformity, minimal ring topology, maximal ring topology, nonideal topology, sigma-bounded ring topology, lattice of ring topologies.

[1]The author was formerly known as Niel Shilkret.

the unique strongest non-Archimedean topology on the rational numbers Q: all nonzero subgroups of Z are $\mathfrak{T}_Z$-open; hence, all nonzero subgroups of Q are $\mathfrak{T}_Z$-open. The topology $\mathfrak{T}_Z$ is also a maximal nondiscrete locally bounded ring topology (see [4]). It is natural to ask if $\mathfrak{T}_Z$ is a maximal topology.

LEMMA 2.1. *If $\mathfrak{B}$ is a neighborhood base at zero of a ring topology on Z, then $\{mV \mid m \in Z^+, V \in \mathfrak{B}\}$ is a subbase at zero of a (possibly discrete) ring topology $\mathfrak{T}$ on Q; $\mathfrak{T}_Z \subset \mathfrak{T}$.*

PROOF. If $U_i - U_i \subset V_i$, then

$$\left(\bigcap m_i U_i\right) - \left(\bigcap m_i U_i\right) \subset \left(\bigcap m_i V_i\right).$$

If $aU_i \subset V_i$, then

$$\frac{a}{b}\left(\bigcap b m_i U_i\right) \subset \left(\bigcap m_i V_i\right).$$

If $W_i + \cdots + W_i \subset V_i$, where there are m_i summands W_i on the left side of the containment, and $U_i^2 \subset W_i$, then

$$\left(\bigcap m_i U_i\right)^2 \subset \left[\bigcap m_i (m_i W_i)\right]$$

$$\subset \left[\bigcap m_i (W_i + \cdots + W_i)\right] \subset \left(\bigcap m_i V_i\right).$$

Since Z is a $\mathfrak{T}$-neighborhood of zero, so are the sets in the $\mathfrak{T}_Z$-base $\{nZ\}_{n \in Z^+}$; therefore, $\mathfrak{T}_Z \subset \mathfrak{T}$.

THEOREM 2. *Let $\{a_1, a_2, \ldots\}$ be a sequence of integers greater than two such that, for each $n \in Z^+$,*

(i) $n! \mid a_n$,

(ii) $a_n \mid a_{n+1}$,

(iii) $a_{n+1} > (2na_n + 1)^{2^{n+1}}$. *If*

If

$$V_{r,n} = \left\{x \in Z \mid x \equiv a(2a_n Z) \text{ for some } a \cdot \ni \cdot |a| < \sqrt[2^r]{2a_n - 1}\right\}$$

$$V_r = \bigcap_{n=1}^{\infty} V_{r,n},$$

then $\mathfrak{S} = \{mV_r \mid m, r \in Z^+\}$ is a neighborhood subbase at zero of a nondiscrete ring topology $\mathfrak{T}$ on Q which is strictly finer than $\mathfrak{T}_Z$.

PROOF. Lemma 2.1 will be applied to show that $\mathfrak{S}$ determines a ring topology on Q at least as fine as $\mathfrak{T}_Z$. Clearly $\{V_r\}$ is a decreasing sequence of symmetric sets containing zero. Let $c = (2a_n)^{2^r}$. Suppose $x, y \in V_{r+1}$ and

$$x \equiv a(2a_n Z), \qquad |a| < \sqrt{c} - 1,$$

$$y \equiv b(2a_n Z), \qquad |b| < \sqrt{c} - 1.$$

If $c < 4$, then $a = b = 0$, which implies $|a + b| < c - 1$; if $c \geqslant 4$, then $|a + b| < 2\sqrt{c} - 2 < 2\sqrt{c} - 1 \leqslant c - 1$. Therefore, $x + y \in V_r$. Also,

$$|ab| < (\sqrt{c} - 1)^2 = c - (2\sqrt{c} - 1) < c - 1.$$

Therefore $xy \in V_r$. From the containments $V_{r+1} + V_{r+1} \subset V_r$ and $V_{r+1}^2 \subset V_r$, which have just been established, it follows that, for any $k \in Z$,

$$kV_{r+|k|} \subset V_{r+|k|} + \cdots + V_{r+|k|} \subset V_r,$$

where there are $2^{|k|}$ summands $V_{r+|k|}$ in the middle member of this chain of containments. It now follows that $\{V_r\}$ is the neighborhood base at zero of a ring topology on Z.

Next it is shown that $\mathfrak{T} \neq \mathfrak{T}_Z$ by showing that only the zero ideal of Z is contained in V_1. Since $2 \leqslant a_n$, for all n, $\sqrt{2a_n} \leqslant a_n$. Since $\pm a_n$ are the representatives of a_n modulo $2a_nZ$ with smallest absolute value, $a_n \notin V_{n,1} \supset V_1$. But, since $n!|a_n$, the sequence $\{a_n\}$ is eventually in every nonzero ideal of Z.

Finally it is shown that $\mathfrak{T}$ is not discrete by observing that

$$2a_R \prod_{i=1}^{t} m_i \in \bigcap_{i=1}^{t} m_i V_r,$$

for $R \geqslant \max(r, \Pi_{i=1}^{t} m_i)$. Let $x = 2a_R \Pi m_i$. Since $a_n | x/m_i$ if $n < R$, $x/m \in V_{r,n}$ for $n \leqslant R$. If $n > R$, then

$$0 \leqslant \frac{x}{m_i} \leqslant 2Ra_R < \sqrt[2^{R+1}]{a_{R+1}} - 1 \leqslant \sqrt[2^r]{a_n} - 1.$$

where the third inequality is a consequence of property (iii) of $\{a_r\}$.

The proof of Theorem 2 contains an explicit description of a nondiscrete ring topology on Z which is stronger than the topology with all nonzero ideals as a basis at zero. The existence of "nonideal" topologies was established in [2].

A subset of a topological ring will be called *σ-bounded* if it can be written as a countable union of bounded sets; a ring topology will be called σ-bounded if the entire ring is a σ-bounded set. Since finite sets are bounded in any ring topology, all ring topologies on a countable ring are σ-bounded.

THEOREM 3. *Any nontrivial σ-bounded ring topology on a ring R can be weakened to a nontrivial first countable ring topology.*

PROOF. Suppose $R = \bigcup_{n=1}^{\infty} B_n$, where $\{B_n\}$ is an increasing sequence of sets bounded in topology $\mathfrak{T}$. Let V_1 be any neighborhood of zero distinct from R. Inductively, choose a sequence $\{V_n\}$ of symmetric $\mathfrak{T}$-open neighborhoods of zero such that $V_{n+1} + V_{n+1} \subset V_n$; $V_{n+1}^2 \subset V_n$; $B_n V_{n+1} \subset V_n$ and $V_{n+1} B_n \subset V_n$. Then $\{V_n\}$ is a neighborhood base at zero for a first countable ring topology contained in $\mathfrak{T}$.

COROLLARY 3.1. *All nontrivial σ-bounded minimal topologies on a ring are first countable. In particular, all minimal topologies on a countable field are first countable.*

There are ring topologies on a simple transcendental extension $C(t)$ of the complex field C which induce the usual topology on C; examples of such topologies are given in [6]. Let $\mathfrak{T}$ be such a topology. Suppose there existed a valuation $|\ |$ on $C(t)$ such that the associated value topology $\mathfrak{T}_{|\ |}$ were contained in $\mathfrak{T}$. Then $\mathfrak{T}_{|\ |}$ would induce a ring topology on C which would be contained in the usual topology on C. But the topology induced on C by $\mathfrak{T}_{|\ |}$ would be the valuable (hence nontrivial) topology of the restriction of $|\ |$; and, furthermore, the usual topology on C is minimal. Therefore, $|\ |$ would induce the usual topology on C. But this is a contradiction, since no valuation on $C(t)$ can be Archimedean on C. Hence, a ring topology on $C(t)$ which induces the usual topology on C does not contain a valuable topology. Thus, (at least) one of two interesting situations must occur: either there exist nonvaluable minimal topologies or there exist topologies which contain no minimal topologies.

References

1. J. Heine and S. Warner, *Ring topologies on the quotient field of a Dedekind domain*, Duke Math. J. **40** (1973), 473–486.

2. L. A. Hinrichs, *Integer topologies*, Proc. Amer. Math. Soc. **15** (1964), 991–995.

3. J. O. Kiltinen, *Inductive ring topologies*, Trans. Amer. Math. Soc. **134** (1968), 149–169.

4. M. E. Shanks and S. Warner, *Topologies on the rational field*, Bull. Amer. Math. Soc. **79** (1973), 1281–1282.

5. N. Shilkret, *Non-Archimedean orthogonality*, Arch. Math. **27** (1976), 67–78.

6. J. H. Williamson, *On topologizing the field $C(t)$*, Proc. Amer. Math. Soc. **5** (1954), 729–734.

Department of Mathematics, City College of New York (CUNY), New York, New York 10031