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RESIDUAL LINEARITY FOR CERTAIN NILPOTENT GROUPS

P. MENAL

ABSTRACT. In this note we consider relations between residual finiteness and residual linearity for a nilpotent group G. We show, amongst other things, that if the center and the commutator subgroup of G are finitely generated and G is residually linear, then G is residually finite. Indeed the property which we use on linear groups is that linear groups satisfy the minimal condition on centralizers.

Let $\mathcal{P}$ be a group property. We recall that a given group G is called residually $\mathcal{P}$ if it is a subdirect product of groups having the property $\mathcal{P}$. For any group G let $R(G)$ be the intersection of all its normal subgroups of finite index. Thus $R(G) = \langle 1 \rangle$ if and only if G is residually finite. For any integer n let G^n be the subgroup of G generated by the nth powers of elements of G. For abelian groups it is well known that $R(G) = \bigcap_{n \geqslant 1} G^n$; if in addition the p-torsion of G is bounded for each prime p, then $R(G)$ is a radicable group. We say that G is residually linear if for each $1 \neq x \in G$ there exists a field K and a homomorphism ϕ: $G \to \mathrm{GL}(n, K)$ such that $\phi(x) \neq 1$. An abelian group is $\mathfrak{R}$ if it is a subdirect product of cyclic groups C_i such that $C_i \cong \mathbf{Z}$ or $|C_i| \leqslant n$ for a fixed integer n. We will use the symbols $\Gamma_n(G)$ and $Z_n(G)$ for the terms in the lower and upper central series of G. If X is a subset of the group G we denote by $C_G(X)$ its centralizer.

The main result of this paper is

THEOREM I. (i) *Let G be a nilpotent residually linear group. If $\Gamma_2(G)$ is finitely generated and $Z_1(G)$ is $\mathfrak{R}$, then G is residually finite.*

(ii) *There exists a nilpotent group of class 2 with $\Gamma_2(G)$ finitely generated and $Z_1(G)$ residually finite, such that it is residually linear but it is not residually finite.*

(iii) *There exists a nilpotent group of class 3 with $Z_1(G)$ cyclic, which is residually linear but it is not residually finite.*

COROLLARY. *Let G be a nilpotent group of class 2 with $Z_1(G)$ finitely generated. Then residually linear implies residually finite.*

LEMMA I. *Let G be a nilpotent linear group. If H is a normal subgroup of G such that $H \cap Z_1(G)$ is finitely generated, then H is finitely generated.*

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Proof. We proceed by induction on the class c of G, the case $c = 1$ being obvious. Suppose that the elements $x_1, x_2, \ldots, x_n$ of G span G linearly, clearly

$$Z_1(G) = C_G(x_1, x_2, \ldots, x_n).$$

Let $[x, y] = x^{-1}y^{-1}xy$ denote commutators in G, then the map

$$H \cap Z_2(G) \to (H \cap Z_1(G)) \times \cdots \times (H \cap Z_1(G))$$

in which $x \mapsto ([x, x_1], \ldots, [x, x_n])$ is a group homomorphism with kernel $H \cap Z_1(G)$. Thus $(H \cap Z_2(G))/(H \cap Z_1(G))$ is finitely generated. Since G is a linear group, $G/Z_1(G)$ is linear [**5**, Theorem 6.2] and we have that

$$(HZ_1(G)/Z_1(G)) \cap Z_1(G/Z_1(G)) \cong (H \cap Z_2(G))/(H \cap Z_1(G))$$

is finitely generated. By induction it follows that

$$HZ_1(G)/Z_1(G) \cong H/(H \cap Z_1(G))$$

is finitely generated and the result is clear.

Lemma 2. *Let G be a nilpotent linear group. Then the following are equivalent.*

(i) *If $x \in G$, the normal closure of $\langle x \rangle$ in G is finitely generated.*

(ii) *$\Gamma_2(G)$ is finitely generated.*

(iii) *$G/Z_1(G)$ is finitely generated.*

Proof. For arbitrary nilpotent groups (iii) implies (ii) [**3**, Corollary 3.19]. Trivially (ii) implies (i). Let $x_1, x_2, \ldots, x_n$ be elements of G spanning G linearly. If we suppose that the normal closure F in G of $\langle x_1, x_2, \ldots, x_n \rangle$ is finitely generated, the homomorphism

$$Z_2(G) \to F \times \cdots \times F$$

in which $x \mapsto ([x, x_1], \ldots, [x, x_n])$ proves that $Z_1(G/Z_1(G))$ is finitely generated. It follows from Lemma 1 that $G/Z_1(G)$ is finitely generated. This proves that (i) implies (iii).

Lemma 3. *Let G be a nilpotent group such that $G/Z_1(G)$ is finitely generated. Then G is residually finite if and only if $Z_1(G)$ is residually finite.*

Proof. It suffices to show that $R(Z_1(G)) = R(G)$. Trivially $R(Z_1(G)) \subseteq R(G)$. For if N is a subgroup of $Z_1(G)$ of finite index, $N \lhd G$, so G/N is finitely generated, nilpotent and hence residually finite [**2**, Theorem 2.1]. Thus $R(G) \subseteq N$ and $R(G) \subseteq R(Z_1(G))$.

We remark that Lemma 3 is a trivial consequence of [**4**, Proposition 1]. However, the above is quite sufficient for our purposes.

Proposition 4. *Let G be a residually linear nilpotent group satisfying the following conditions.*

(i) *If $x \in G$, the normal closure of $\langle x \rangle$ in G is finitely generated.*

(ii) *$G/\Gamma_2(G)$ is residually finite and for each prime p its p-torsion is bounded.*

Then G is residually finite.

PROOF. Let $1 \neq x \in G$. We will prove that $x \notin R(G)$. Since G is residually linear we can consider a homomorphism ϕ of G into a linear group such that $\phi(x) \neq 1$. Let $\bar{G} = G/(\mathrm{Ker}\,\phi \cap \Gamma_2(G))$. Since homomorphic images of G satisfy (i) it follows from Lemma 2 that $\Gamma_2(G/\mathrm{Ker}\,\phi)$ and $(G/\mathrm{Ker}\,\phi)/Z_1(G/\mathrm{Ker}\,\phi)$ are finitely generated. Clearly $\bar{G} \hookrightarrow (G/\mathrm{Ker}\,\phi) \times (G/\Gamma_2(G))$. Then we see easily that $\Gamma_2(\bar{G})$ and $\bar{G}/Z_1(\bar{G})$ are finitely generated. Furthermore we have

$$\bar{G}/\Gamma_2(\bar{G}) \cong G/\Gamma_2(G).$$

Thus by (ii) we conclude that the p-torsion of $\bar{G}$ is bounded for each prime p. Therefore $R(Z_1(\bar{G}))$ is a radicable group. But $\bar{G}/\Gamma_2(\bar{G})$ is residually finite so $R(Z_1(\bar{G})) \subseteq \Gamma_2(\bar{G})$. Since $\Gamma_2(\bar{G})$ is finitely generated, necessarily $R(Z_1(\bar{G})) = \langle 1 \rangle$. Now Lemma 3 implies that $\bar{G}$ is residually finite. Since $x \notin \mathrm{Ker}\,\phi \cap \Gamma_2(G)$ we conclude that $x \notin R(G)$.

LEMMA 5. *Let G be a $\mathfrak{R}$ group and let H be a finitely generated subgroup. Then G/H is a $\mathfrak{R}$ group.*

PROOF. G is a $\mathfrak{R}$ group hence $G \subseteq \prod \mathbf{Z} \times C$. Where C is a bounded group and so a direct sum of cyclic groups [**1**, Theorem 17.2]. Since subgroups of $\mathfrak{R}$ groups are $\mathfrak{R}$ groups, we can assume $G = \prod \mathbf{Z} \times C$ in order to prove the lemma. Let H be a finitely generated subgroup of G. Then there exist finitely generated subgroups $M \subseteq \prod \mathbf{Z}$ and $N \subseteq C$ such that $H \subseteq M \times N$. Every finitely generated subgroup of $\prod \mathbf{Z}$ can be embedded in a finitely generated direct summand of $\prod \mathbf{Z}$ [**1**, Theorem 19.2]. Clearly the same property holds for C. Therefore we may, in addition, suppose that $M \times M' = \prod \mathbf{Z}$ and $N \times N' = C$ for some $M' \subseteq \prod \mathbf{Z}$ and $N' \subseteq C$. Now we have

$$G/H \cong (M \times N/H) \times M' \times N'$$

and the result follows.

We can now give the

PROOF OF THEOREM I. (i) By the proposition, we have only to prove that $G/\Gamma_2(G)$ is residually finite and the elements of $G/\Gamma_2(G)$ of finite order are of bounded order. We use induction on the class c of G. If $c = 1$ the result is trivial. Suppose $c > 1$ and let $\bar{G} = G/Z_1(G)$. Trivially $\Gamma_2(\bar{G})$ is finitely generated. $Z_1(\bar{G})$ is $\mathfrak{R}$, since it is contained in a product $\prod Z_1(G)$. By induction we have that $\bar{G}/\Gamma_2(\bar{G})$ is residually finite and its torsion is bounded. Since $Z_1(G)$ is $\mathfrak{R}$ and $\Gamma_2(G)$ is finitely generated it follows from Lemma 5 that $Z_1(G)\Gamma_2(G)/\Gamma_2(G)$ is $\mathfrak{R}$. Therefore we have that $G/\Gamma_2(G)$ is of torsion bounded. Thus $R(G/\Gamma_2(G))$ is a radicable group contained in $Z_1(G)\Gamma_2(G)/\Gamma_2(G)$. Since $\mathfrak{R}$ groups contain no nontrivial radicable groups we conclude that $G/\Gamma_2(G)$ is residually finite.

(ii) Let p be a prime. Put

$$G = \langle z_i, x_i, y_i, i = 0, 1, \ldots : z_{i+1}^p = z_i, \\ [x_i, x_j] = [y_i, y_j] = [z_i, z_j] = [z_i, x_j] = [z_i, y_j] = 1, \\ [x_i, y_j] = 1 \text{ if } i \neq j, [x_i, y_i] = z_0^{p^i} \rangle.$$

G is a nilpotent group of class 2 with $\Gamma_2(G) = \langle z_0 \rangle$ and

$$Z_1(G) = \langle z_i, i = 0, 1, \dots \rangle \cong Q_p$$

(where Q_p is the group of all rational numbers whose denominators are powers of p). $Z_1(G)$ is residually finite however it does not satisfy $\mathfrak{R}$. We will prove that G is not residually finite but it is residually linear. Suppose that $x \mapsto \bar{x}$ is a homomorphism of G into a finite group $\bar{G}$. Then, by the finiteness of $\bar{G}$, there exist distinct integers n, m with $\bar{x}_n = \bar{x}_m$. Thus $\bar{1} = [\bar{x}_m, \bar{y}_m] = [\bar{x}_n, \bar{y}_m] = \bar{z}_0^{p^m}$. Since a finite homomorphic image of Q_p has no elements of order p, we have that $\bar{z}_0 = \bar{1}$ so $z_0 \in R(G)$. In fact $R(G) = Z_1(G)$. Let K be a field containing the p^n-roots of the unity for any integer $n \geqslant 1$. In order to prove that G is residually linear it suffices to show that the group $G_n = G/\langle z_0^{p^n} \rangle$ is residually K-linear for any integer $n \geqslant 1$, since $\bigcap_{n \geqslant 1} \langle z_0^{p^n} \rangle = \langle 1 \rangle$. It follows from the relations of G that $Z_1(G_n)$ is $\mathbf{Z}(p^\infty)$ by a residually finite group. Therefore $Z_1(G_n)$ is residually K-linear. Clearly $Z_1(G_n)$ has finite index in G_n so G_n is residually K-linear and the result follows.

(iii) Let p be a prime. Let G be a group generated by z, t_i, x_i, y_i, $i = 1, 2, \dots$, subject to the relations

$$[x_i, x_j] = [y_i, y_j] = [t_i, t_j] = [z, x_i] = [z, t_i] = [z, y_i] = 1,$$

$$[x_i, y_i] = t_i^{p^i} z, \qquad [x_i, y_j] = 1 \quad \text{if } i \neq j,$$

$$[t_i, x_i] = [t_i, y_i] = z^{p^i}, \quad [t_i, x_j] = [t_i, y_j] = 1 \quad \text{if } i \neq j.$$

G is a torsion free nilpotent group of class 3 with center $\langle z \rangle$. Let $x \mapsto \bar{x}$ be a homomorphism of G into a finite group $\bar{G}$. Then, by finiteness of $\bar{G}$, there exist distinct integers n, m with $\bar{y}_n = \bar{y}_m$. Thus

$$\bar{1} = [\bar{t}_m, \bar{y}_n] = [\bar{t}_m, \bar{y}_m] = \bar{z}^{p^m}.$$

Suppose that $\bar{z} \neq \bar{1}$. Then $h_p(\bar{z})$, the p-height of $\bar{z}$ in $\bar{G}$, is finite. Again there exist distinct integers $r, s > h_p(\bar{z})$, with $\bar{1} = [\bar{x}_r, \bar{y}_s] = [\bar{x}_s, \bar{y}_s] = \bar{t}_s^{p^s} \bar{z}$. Hence $\bar{z} = (\bar{t}_s^{-1})^{p^s}$ and $s \leqslant h_p(\bar{z})$, a contradiction. Therefore we have shown that z belongs to the kernels of all homomorphism of G into finite groups so $z \in R(G)$. Finally we show that G is residually K-linear, if K contains, for every n, the p^n-roots of the unity. Define for each integer $n \geqslant 1$

$$H_n = \langle t_1^{p^n}, t_2^{p^n}, \dots, t_{n-1}^{p^n}, z^{p^n}, t_m^{p^m} z \text{ for } m \geqslant n \rangle.$$

Clearly H_n is a normal subgroup of G and $H_n \cap \langle z \rangle = \langle z^{p^n} \rangle$. Therefore $\bigcap_{n \geqslant 1} H_n = \langle 1 \rangle$. Then it suffices to prove that the group $\bar{G} = G/H_n$ is residually K-linear. It is clear that

$$\bar{G} = \langle Z_1(\bar{G}), \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}_1, \dots, \bar{y}_{n-1}, \bar{t}_1, \dots, \bar{t}_{n-1} \rangle.$$

Furthermore, for $i = 1, 2, \dots, n-1$, we have

$$\bar{t}_i^{p^n} = \bar{1} \quad \text{so } [\bar{x}_i^{p^n}, \bar{t}_i] = [\bar{y}_i^{p^n}, \bar{t}_i] = \bar{1},$$

$$[\bar{x}_i^{p^n}, \bar{y}_i] = [\bar{x}_i, \bar{y}_i]^{p^n} [\bar{x}_i, \bar{y}_i, \bar{x}_i]^{p^n(p^n-1)/2}$$

$$= \bar{t}_i^{p^{i+n}} \bar{z}^{p^n} \bar{z}^{p^{2i+n}(p^n-1)/2} = \bar{1},$$

similarly $[\bar{y}_i^{p^n}, \bar{x}_i] = \bar{1}$. These relations yield that $\bar{G}/Z_1(\bar{G})$ is a torsion group which is finite, since it is finitely generated. The result follows, since $Z_1(\bar{G})$ is residually K-linear.

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SECCIÓ DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, BARCELONA, ESPAÑA