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ON A NECESSARY CONDITION FOR THE ERDÖS-RÉNYI LAW OF LARGE NUMBERS

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ABSTRACT. For a sequence $\{X_i\}_{i=1,2,\ldots}$ of independent, identically distributed random variables with existing moment-generating function $\varphi(t) = E\exp(tX_i)$ in some nondegenerate interval, Erdös and Rényi (1970) studied the maximum $D(N, K)$ of the $N - K + 1$ sample means $K^{-1}(S_{n+K} - S_n)$, $0 \leq n \leq N - K$, where $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. They showed that for a certain range of numbers $\mathbf{a}$ there exist positive constants $C(\mathbf{a})$ such that $\lim_{N\to\infty} D(N, [C(\mathbf{a})\log N]) = \mathbf{a}$ with probability one. In the present paper it is shown that the existence of the moment-generating function is also a necessary condition, i.e. that $\limsup_{N\to\infty} D(N, [C\log N]) = \infty$ for every positive constant C, if the moment-generating function does not exist for any positive number t.

In 1970, Erdös and Rényi developed what they called 'a new law of large numbers'. In the general case this law makes the following assertion:

THEOREM 1 (ERDÖS-RÉNYI). *Let $\{X_i\}_{i=1,2,\ldots}$ be a sequence of independent, identically distributed (i.i.d.) nondegenerate random variables on a probability space $(\Omega, \mathfrak{A}, P)$. Suppose that*

$$\varphi(t) = Ee^{tX_1} < \infty \quad \text{for every } t \text{ in some interval } (0, T). \tag{1}$$

Let $\mathbf{a}$ be any real number such that the function $\varphi(t)e^{-t\mathbf{a}}$ takes on its minimum value in the interval $(0, T)$ and put

$$\min_{t\in(0,T)} \varphi(t)e^{-t\mathbf{a}} = e^{-1/C}.$$

Then $C > 0$, and putting $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, and

$$D(N, K) = \max_{0\leq n\leq N-K} \frac{S_{n+K} - S_n}{K} \qquad (1 \leq K \leq N),$$

it follows that

$$\lim_{N\to\infty} D(N, [C\log N]) = \mathbf{a} \tag{2}$$

with probability one. Here $[x]$ denotes the integer part of x.

PROOF. See [5, Theorem 2].

REMARK. Erdös and Rényi assumed that the moment-generating function

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$\varphi(t)$ exists in some open interval I containing $t = 0$, and put $EX_i = 0$ without loss of generality. However, the weaker assumption (1) is sufficient for proving (2), since (1) already yields an exponential convergence rate for the probabilities $P(S_N \geqslant N\mathbf{a})$ (cf. [7]). The latter was an essential tool in Erdös' and Rényi's proof. Thus, the Erdös-Rényi law of large numbers may hold even if the expectation of the X_i does not exist.

As mentioned above the proof of Theorem 1 is mainly based on the existence of the moment-generating function of the X_i in a nondegenerate interval which yields exponential large deviation rates. Other versions of the Erdös-Rényi law of large numbers given by Book [2], [3], [4] also make use of moment-generating function techniques. Now, it is well known that, under certain conditions, the existence of the moment-generating functions of the underlying random variables is even necessary to retain exponential convergence rates (cf. [1] and [6]). Therefore, the close connection between the Erdös-Rényi law of large numbers and exponential large deviation probabilities raises the question whether assumption (1) in Theorem 1 is also necessary to retain assertion (2). Using a result of Petrov and Širokova (1973) we are able to give a positive answer.

THEOREM 2. *Let* $\{X_i\}_{i=1,2,\ldots}$ *be a sequence of i.i.d. random variables with*

$$\varphi(t) = Ee^{tX_i} = \infty \quad \text{for all } t > 0. \tag{3}$$

Then, using the notations of Theorem 1, *it follows that*

$$\limsup_{N\to\infty} D(N, [C \log N]) = \infty \tag{4}$$

with probability one for every positive constant C.

The proof of Theorem 2 is based on the following result in [6] which is a one-sided analogue to the lemma in Chapter 2 of [1].

LEMMA. *Let* $\{X_i\}_{i=1,2,\ldots}$ *be a sequence of i.i.d. random variables with*

$$P(S_N \geqslant N\mathbf{a}) \leqslant A\rho^N, \qquad N = 1, 2, \ldots,$$

for some constants $\mathbf{a}$, A, *and* $\rho < 1$. *Then there exists a positive real number* T *such that*

$$\varphi(t) = Ee^{tX_1} < \infty \quad \text{for all } t \in [0, T].$$

PROOF. See [6, Theorem 1].

From the above lemma we obtain an immediate corollary which is required for the proof of Theorem 2.

COROLLARY. *Let* $\{X_i\}_{i=1,2,\ldots}$ *be a sequence of i.i.d. random variables such that* (3) *holds. Then it follows that*

$$\limsup_{N\to\infty} \frac{P(S_N \geqslant N\mathbf{a})}{\rho^N} = \infty \tag{5}$$

for all constants $\mathbf{a}$ *and* ρ, *where* $0 < \rho < 1$.

We will now turn to the proof of the main result.

Proof of Theorem 2. For arbitrary **a** and $\rho < 1$, (5) implies the existence of a subsequence $\{N_k\}_{k=1,2,\ldots}$ of natural numbers with

$$P(S_{N_k} \geqslant N_k \mathbf{a}) \geqslant \rho^{N_k}, \qquad k = 1, 2, \ldots.$$

Let $C > 0$ be fixed. Then the sequence $\{N_k\}_{k=1,2,\ldots}$ can be chosen such that

$$N_k = [C \log N_k'], \qquad k = 1, 2, \ldots,$$

for another subsequence $\{N_k'\}_{k=1,2,\ldots}$ of natural numbers. Note that $\lim_{N\to\infty}\{C \log(N+1) - C \log N\} = 0$. Put $\rho = \exp(-1/C')$, where $C' > C$. Now,

$$\begin{aligned} P\big(D(N_k', [C \log N_k']) < \mathbf{a}\big) &= P(D(N_k', N_k) < \mathbf{a}) \\ &\leqslant P\left(\bigcap_{i=1}^{[N_k'/N_k]} \{S_{iN_k} - S_{(i-1)N_k} < N_k \mathbf{a}\}\right) \\ &= \{1 - P(S_{N_k} \geqslant N_k \mathbf{a})\}^{[N_k'/N_k]} \\ &\leqslant \{1 - \rho^{N_k}\}^{[N_k'/N_k]} \leqslant \exp\big(-\rho^{N_k}[N_k'/N_k]\big). \end{aligned}$$

Using $C' > C$, we have

$$\rho^{N_k} = \rho^{[C \log N_k']} \geqslant \rho^{C \log N_k'} = N_k'^{-C/C'} = N_k'^{-(1-2\delta)}$$

for some $\delta > 0$. Furthermore,

$$[N_k'/N_k] = [N_k'/[C \log N_k']] \geqslant N_k'^{1-\delta}$$

for all sufficiently large k, say $k \geqslant k_0$. Hence it follows that

$$P\big(D(N_k', [C \log N_k']) < \mathbf{a}\big) \leqslant \exp(-N_k'^{\delta})$$

for $k \geqslant k_0$, and

$$\sum_{k=k_0}^{\infty} P\big(D(N_k', [C \log N_k']) < \mathbf{a}\big) \leqslant \sum_{k=k_0}^{\infty} \exp(-N_k'^{\delta}).$$

The last series converges by the integral test. Thus, the Borel-Cantelli lemma yields

$$\liminf_{k\to\infty} D(N_k', [C \log N_k']) \geqslant \mathbf{a}$$

with probability one, and, moreover,

$$\limsup_{N\to\infty} D(N, [C \log N]) \geqslant \mathbf{a}$$

with probability one. Since **a** can be chosen arbitrarily large, (4) is proven.

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