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ON THE BOUNDARY VALUE PROBLEM $u'' + u = \alpha u^- + p(t), u(0) = 0 = u(\pi)$

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ABSTRACT. In this note we consider the boundary value problem $u'' + u = \alpha u^- + p(t)$, $u(0) = 0 = u(\pi)$, $\alpha > 0$, and show that a necessary and sufficient condition for the problem to be solvable is that $\int_0^\pi p(s) \sin s \, ds < 0$. We thus answer in the affirmative a question posed by S. Fučik.

Consider the boundary value problem

$$u'' + u = \alpha u^- + p(t), \qquad u(0) = 0 = u(\pi), \tag{1}$$

where $u^-(t) = \max\{-u(t), 0\}$, α is a positive constant and p: $[0, \pi] \to R$ is a continuous function such that

$$\int_0^\pi p(s) \sin s \, ds \leqslant 0. \tag{2}$$

In [1] Fučik poses the question as to whether (2) is a necessary and sufficient condition in order that (1) have a solution for any $\alpha > 0$.

In this note we answer the question in the affirmative.

Our proof utilizes a continuation theorem of Mawhin [2] which we state here for the reader's convenience.

The necessary notation is the following:

X and Y are real Banach spaces, L: $\operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero, N: $\bar{\Omega} \subset X \to Y$ is a not necessarily linear operator and Ω is a bounded open subset of X. P: $X \to X$, Q: $Y \to Y$ denote bounded linear projections such that $\operatorname{im} P = \ker L$, $\operatorname{im} L = \ker Q$ and $K_{P,Q} = (L|\ker P \cap \operatorname{dom} L)^{-1}(I - Q)$, where I is the identity map on Y. N is assumed to be L-compact, i.e., QN: $\bar{\Omega} \to Y$ is continuous, $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N$: $\bar{\Omega} \to X$ is completely continuous.

In this setting the following continuation theorem holds [2].

THEOREM 1. *Assume*:

(3) $\forall \lambda \in (0, 1), \forall x \in \operatorname{dom} L \cap \partial\Omega, Lx \neq \lambda Nx$.

(4) $\forall x \in \ker L \cap \partial\Omega, Nx \notin \operatorname{im} L$, *i.e.*, $QNx \neq 0$.

(5) $d[JQN|\ker L, \Omega \cap \ker L, 0] \neq 0$, *where d denotes the Brouwer degree and J*: $\operatorname{im} Q \to \ker L$ *is some isomorphism*.

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Then there exists $x \in \bar{\Omega} \cap \operatorname{dom} L$ *such that*

(6) $$Lx = Nx.$$

With the aid of Theorem 1 we shall now establish the following:

THEOREM 2. *Let* $p\colon [0, \pi] \to R$ *be continuous. Then* (1) *has a solution for any* $\alpha > 0$ *if and only if* (2) *holds.*

PROOF. We multiply the equation by $\sin t$, integrate from 0 to π, and obtain that the condition is necessary.

If $\int_0^\pi p(s) \sin s\, ds = 0$ there exists a solution u of $u'' + u = p(t)$, $u(0) = 0 = u(\pi)$ such that $u(t) > 0$, $0 < t < \pi$; thus to prove the sufficiency it suffices to assume that

(7) $$\int_0^\pi p(s) \sin s\, ds < 0.$$

In order to be able to apply Theorem 1 we let

$$X = C_0^2[0, \pi] = \{x \in C^2[0, \pi] \colon x(0) = 0 = x(\pi)\}, \quad Y = C[0, \pi],$$

both spaces being equipped with their usual maximum norms which are denoted by $|\cdot|_2$ and $|\cdot|$, respectively. We let $L\colon X \to Y$ be defined by $u \mapsto u'' + u$ and $N\colon X \to Y$ by $u \mapsto \alpha u^- + p$ where $\alpha > 0$ is fixed.

Define the projections $P\colon X \to X$ and $Q\colon Y \to Y$ by

$$(Px)(t) = \frac{\alpha \sin t}{\pi} \int_0^\pi x(s) \sin s\, ds, \qquad x \in X,$$

$$(Qx)(t) = \frac{\alpha \sin t}{\pi} \int_0^\pi x(s) \sin s\, ds, \qquad x \in Y.$$

Then one easily calculates that $\operatorname{im} P = \ker L$ and $\operatorname{im} L = \ker Q$. Furthermore, $K_{P,Q}N$ is completely continuous (it is a composition of a compact linear mapping with a continuous operator).

In order to be able to apply Theorem 1 we show that there exists $r > 0$ such that if u is a solution of

(8) $$Lu = \lambda N u, \qquad 0 < \lambda < 1,$$

then $|u| < r$; this will imply by standard arguments that there exists $r_1 > r$ such that $|u|_2 < r_1$. We then show that Ω may be chosen as $\{u\colon |u|_2 < r_2\}$, where r_2 is some suitably chosen constant larger than r_1.

To accomplish this we make some auxiliary considerations. For $0 < a < \pi$, $0 < \lambda < 1$ consider the initial value problems

$$v'' = v + \lambda p(t), \qquad v(a) = 0, \qquad v'(a) = \beta.$$

Its solution is given by

$$v(t, \beta) = \beta \sin(t - a) + \lambda \sin t \int_a^t p(s) \cos s\, ds$$
$$- \lambda \cos t \int_a^t p(s) \sin s\, ds.$$

This, via an easy calculation, implies the following:

(9) $\exists\, \beta^*$ (independent of λ) such that if $1/\sqrt{1+\alpha} \leqslant a \leqslant \pi - 1/\sqrt{1+\alpha}$, then $v(t, \beta^*) > 0, a < t < \pi, v(t, -\beta^*) < 0, 0 < t < \alpha$.

Let u be a solution of (8). Then since (7) holds it cannot be the case that $u(t) > 0, 0 < t < \pi$. Thus either there exists $a \in (0, \pi)$ such that $u(a) = 0$ or $u(t) < 0$ for all t with $0 < t < \pi$. Consider solutions of (8) that satisfy the former.

If $a \in [1/\sqrt{1+\alpha}, \pi - 1/\sqrt{1+\alpha}]$ and $u'(a) > 0$, then $u(t) \leqslant v(t, \beta^*)$, where β^* and $v(t, \beta^*)$ are given by (9); thus $u'(a) \leqslant \beta^*$; if, however, $u'(a) < 0$, then $u(t) \leqslant v(t, -\beta^*)$, $0 \leqslant t \leqslant a$ and, hence, $-\beta^* \leqslant u'(a)$. (These facts follow easily from the observation that solutions of $v'' + v = \lambda p(t)$ may cross at most once on any interval of length less than π.) From this it follows that $|u'(a)| \leqslant \beta^*$.

Next consider the cases that $a \in (0, 1/\sqrt{1+\alpha})$ and $a \in (\pi - 1/\sqrt{1+\alpha}, \pi)$. We treat the former, the latter being not much different. Thus u will be a solution of the boundary value problem

$$u'' + u = \lambda\alpha u^- + \lambda p, \qquad u(0) = 0 = u(a)$$

and, hence, satisfies the operator equation

$$u(t) = \int_0^a G(t,s)\left[u(s) - \lambda\alpha u^-(s) - \lambda p(s)\right] ds, \qquad 0 \leqslant t \leqslant a,$$

where G is the Green's function associated with the operator $D^2 u = u''$.

Hence (letting $\|u\| = \max_{[0,a]}|u(t)|$),

$$|u(t)| \leqslant (a^2/8)(1+\alpha)\|u\| + (a^2/8)|p|,$$

and thus

$$\|u\| \leqslant |p|/7(1+\alpha).$$

The integral equation therefore implies

$$|u'(a)| \leqslant \frac{a}{2}\,[1+\alpha]\|u\| + \frac{a}{2}\,|p| \leqslant \frac{4}{7\sqrt{1+\alpha}}\,|p|.$$

The same bound will hold in case $a \in (\pi - 1/\sqrt{1+\alpha}, \pi)$. Hence $|u'(a)| \leqslant \delta$, where $\delta = \max\{\beta^*, 4|p|/7(\sqrt{1+\alpha})\}$. Let $\mu = u'(a)$; then u is the solution of the initial value problem

$$u'' + u = \lambda\alpha u^- + \lambda p(t), \qquad u(a) = 0, \qquad u'(a) = \mu.$$

Let this solution be denoted by $u(t, a, \mu, \lambda)$. It follows from elementary continuous dependence results that the mapping $(a, \mu, \lambda) \mapsto u(\cdot, a, \mu, \lambda)$ is a continuous mapping from R^3 to Y.

Hence the set of solutions of the boundary value problems considered is a subset of the family $\{u(\cdot, a, \mu, \lambda)\colon 0 \leqslant a \leqslant \pi,\ 0 \leqslant \lambda \leqslant 1,\ |\mu| \leqslant \delta\}$. This family is precompact in $C[0, \pi]$, hence it follows that there exists $r > 0$ (independent of λ) such that any such solution u of (8) satisfies $|u| < r$ and, hence, there exists $r_1 > r$ such that $|u|_2 < r_1$.

We next show that there exists $r_3 > 0$ independent of λ such that if $u(t)$ is a solution to (8) with $u(t) < 0$ for all $t \in (0, \pi)$, then $|u|_2 < r_3$. If $u(t)$ is such a solution, then (8) reduces to $u'' + \beta^2 u = \lambda p$, where $\beta^2 = 1 + \lambda\alpha$. Thus

$$u(t) = A(\lambda)\sin \beta t + \frac{\lambda}{\beta} \sin \beta t \int_0^t \cos \beta s p(s)\, ds \tag{10}$$
$$- \frac{\lambda}{\beta} \cos \beta t \int_0^t \sin \beta s p(s)\, ds.$$

If $\beta^2 \geqslant \frac{9}{4}$, there exists t_λ such that $\sin \beta t_\lambda = -1$, and thus, since $u \leqslant 0$ and$A(\lambda) \leqslant 0$, it follows from (10) that $|A(\lambda)| \leqslant 4\pi|p|/3$. Otherwise we have (since $u(\pi) = 0$) that

$$A(\lambda) = \frac{\lambda}{\sin\sqrt{1 + \lambda\alpha\pi}} \frac{\cos \beta\pi}{\beta} \int_0^\pi \sin \beta s p(s)\, ds \tag{11}$$
$$- \frac{\lambda}{\beta} \int_0^\pi \cos \beta s p(s)\, ds.$$

Using L'Hospital's rule we find that there exists $M > 0$ such that $|A(\lambda)| \leqslant M$ for $0 \leqslant \lambda \leqslant 5/4\alpha$, and hence $|A(\lambda)| \leqslant \max\{4\pi|p|/3, M\}$, $0 < \lambda < 1$. Therefore there exists $r_3 > 0$ such that if u is a solution of (8) with $u(t) < 0$, $0 < t < \pi$, then $|u|_2 < r_3$.

Let $\Omega = \{u \in X\colon |u|_2 < r_2\}$, where $r_2 \geqslant \max\{r_1, r_3\}$ is to be chosen such that (4) and (5) hold.

Let $x \in \ker L \cap \partial\Omega$; then $x(t) = \gamma \sin t$, and

$$(QNx)(t) = \frac{2 \sin t}{\pi} \int_0^\pi \left[(\alpha\gamma \sin s)^- + p(s) \right] \sin s\, ds$$
$$= \alpha \frac{2 \sin t}{\pi} \int_0^\pi (\gamma \sin s)^- \sin s\, ds + \frac{2 \sin t}{\pi} \int_0^\pi p(s) \sin s\, ds.$$

Hence if we let $z = \int_0^\pi p(s) \sin s\, ds$, it follows that $QNx \neq 0$ for $x \in \ker L \cap \partial\Omega$ as long as $\alpha\pi|r_2|/3 \neq |z|$. Thus choose $r_2 > r_1$ such that $r_2 > 3|z|/\alpha\pi$.

We let J: im $Q \to \ker L$ be the identity mapping and observe that, because of the above calculation, $d(JQN|\ker L,\ \Omega \cap \ker L, 0)$ is defined for any $r_2 > 3|z|/\alpha\pi$.

To compute the latter we let T: $\ker L \to R$ be defined by $T(\mu \sin t) = \mu$. Then

$$d\left[QN|\ker L, \Omega \cap \ker L, 0\right] = d\left[T(QN|\ker L)T^{-1}, T(\Omega \cap \ker L), 0\right].$$

Let $\phi(\mu) = (TQN|\ker LT^{-1})(\mu)$; then

$$\phi(\mu) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^\pi p(s) \sin s\, ds & \text{if } \mu > 0, \\ -\alpha\mu + \dfrac{2}{\pi} \displaystyle\int_0^\pi p(s) \sin s\, ds & \text{if } \mu \leqslant 0. \end{cases}$$

On the other hand, since $r_2 > 3|z|/\alpha\pi$, it follows that the only zero of ϕ is contained in $(-r_2, r_2)$. Thus $d(\phi, (-r_2, r_2), 0) = -1$. We have therefore verified all hypotheses of Theorem 1 and thus proved Theorem 2.

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